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ANALYSIS OF
FIBER-REINFORCED
LAMINATED MOMENTUM
WHEEL ROTORS

RANIA ADEL HASSAN

1999
THE AMERICAN UNIVERSITY IN CAIRO
SCHOOL OF SCIENCES AND ENGINEERING
ENGINEERING DEPARTMENT

ANALYSIS OF FIBER-REINFORCED LAMINATED
MOMENTUM WHEEL ROTORS

BY
RANIA ADEL HASSAN

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science in Engineering

with concentration in

Design

under the supervision of

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July 1999
Analysis of Fiber-Reinforced-Laminated Momentum Wheel Rotors

A Thesis Submitted by
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to the Department of Engineering

July 28, 1999

in partial fulfillment of the requirements for the degree of

Master of Science in Engineering with Specialization in Design

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To My Sister Raghdah
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Abstract

The objective of this study is to analyze and optimize the parameters involved in the design of a rotor which is made from graphite-reinforced-epoxy laminated concentric rings with interferences. The proposed rotor design can be used for a Fixed Momentum Wheel (FMW) that is used as a momentum storage device for the attitude control of a medium-size geo-synchronous broadcasting satellite. The study shows that the suggested design concept is also pertinent to other applications.

One of the main motives behind this new design is the success of the development and use of cost-effective magnetic bearings for space applications. Magnetic bearings allow for high rotational speeds with no friction or stiction, and with lower power consumption. For the rotor to sustain the high stresses induced from the centrifugal effect, laminated composite materials are suggested. The study includes a comparison between the performance of conventional isotropic aluminum rotors and the suggested laminated composite rotor to show the advantage of the latter over the former material.

Two types of laminate design are investigated under plane stress conditions. The first is axi-symmetric laminate, and the second is general orthotropic axi-nonsymmetric laminate. To analyze both cases, the formulation of the problem is addressed in the domain of the elasticity theory. In addition, comprehensive models for the estimation of laminated composite materials’ properties are furnished. While the solution obtained for the axi-symmetric laminate is a closed form solution, the one obtained for the axi-nonsymmetric laminate is a numerical solution using the Finite Element Method.

Four parameters of interest are studied for the axi-symmetric laminate design to reach an optimum design for the rotor: the number of concentric rings, the amount of radial interferences between the rings, reinforcement volume fraction of each ring, and the lamination angle of the fibers in the hoop-radial plane with respect to the global hoop direction of the rings.

It was found out that the performance of the suggested graphite-reinforced-epoxy laminated rotor exceeds by far that of the conventional aluminum rotor in terms of
stiffness and strength. Furthermore, the parametric analysis of the axi-symmetric laminated concentric rings rotor shows that the amount of interferences between the neighboring rings should be kept at minimum values just to keep the rings in contact at maximum operating speeds. The analysis also proves that by increasing the amount of reinforcement volume fraction, the minimum interferences needed can be reduced proportionally. For this space application, and with the objective of weight minimization in mind, high speeds are needed to achieve the momentum storage requirement. The analysis shows that at such speeds, the high-stiffness, high-strength reinforcing fibers should be aligned in the circumferential direction to account for the resulting high hoop stresses. Finally, the analysis proves that the number of the rings that the rotor consists of should be kept at a minimum.
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**Abbreviations**

ADCS  
Attitude Determination and Control Subsystem

CTE  
Coefficient of Thermal Expansion

CLT  
Classical Lamination Theory

FMW  
Fixed Momentum Wheel

FEM  
Finite Element Model

GEO  
Geo-stationary(Geo-synchronous) Earth Orbit

GRE  
Graphite-Reinforced-Epoxy

LEO  
Low Earth Orbit

RMW  
Reaction Momentum Wheel

**Nomenclature**

E  
Young’s Modulus

G  
Modulus of Rigidity

K  
Bulk’s Modulus

\( u \)  
radial component of displacement

\( v \)  
hoop component of displacement

\( w \)  
axial component of displacement

\( \nu \)  
Poisson’s Ratio

\( \sigma \)  
Normal Stress

\( \tau \)  
Shear Stress

\( \varepsilon \)  
Normal Strain

\( \gamma \)  
Shear Strain

\( \rho \)  
Density

\( \omega \)  
Angular Velocity
Chapter One

Introduction and Literature Review

1.1 INTRODUCTION

One of the main concerns in the aerospace industry is the weight of space structures and space vehicles’ subsystems and components. The primary reason for the importance of weight minimization, as a design objective, is the prohibitive cost associated with the launching of space structures into their orbital positions. The cost of launching one kg of payload into its orbital position ranges from 22,500 to 30,000 US $, be it on the American, European, Russian or Chinese expendable boosters [1.1]. The cost of launching a geostationary medium size broadcasting satellite of around 1800 kg exceeds 50% of its total cost, including design, manufacturing, assembly, testing and launching [2.1].

Recent statistics show that around ten flights, each carrying from one to three commercial broadcasting satellites, are launched yearly to their orbital position on the geostationary orbit, which is 36,000 km away from Earth’s surface [2.1]. This is of course in addition to the rapidly increasing number of low orbit communication satellites, the most famous of which being the Iridium constellation.

Since the commercial satellite business is obviously going through its most prosperous era and, at the same time, there are many scheduled scientific missions that will be sent to the Space Station regularly, a new, less costly launching technique is certainly needed. Reusable launch vehicles are currently under development, and are expected to cut the launch costs down to about 2250 US $ per kilogram of payload [1.1].

The alternative classical approach regarding the cut down of launch costs, which is the minimization of space structures’ components weight, remains important as it is still needed even with the new reusable vehicles. In general, less weight means less power consumption, which ultimately means less cost. The design of such components attempts to minimize the weight in general, and at the same time meet the specific components’ performance requirements. The use of composite materials is one of the tools that has been utilized extensively to achieve this design objective.
This study is concerned with a fixed momentum wheel rotor design, which is used in attitude control of a medium size geo-synchronous broadcasting satellite. The main function of the wheel is to provide gyroscopic stiffness for the spacecraft as a means of stabilization in a certain plane. It also provides attitude control for the satellite against certain types of external disturbing torques. The classic design of the rotor uses an aluminum alloy, and the maximum speed is 5,060 rpm at a 55 Nms of angular momentum. The limitation on the speed is dictated by the use of angular contact ball bearings that must endure for the overall spacecraft lifetime, which is 15 years. Recently, magnetic bearings were introduced as an alternative to ball bearings. Magnetic bearings can cope with any speed, since there is no contact between the rotor and the stator, and hence there is no friction that would limit the lifetime of the bearings.

With the idea of magnetic bearings in mind, the design approach of the system under consideration would shift towards minimizing the weight of the wheel rotor while using higher speeds to ultimately obtain the required angular momentum, that is the product of the rotor mass moment of inertia and its operational angular velocity. The high speeds develop high stresses and strains, which need special consideration in the design of the rotor. The optimum solution for the low weight specification and the high strains and stresses burden is the use of composite materials in the design of the rotor. The study focuses on the elements of the design of the rotor using composite laminates.
1.2 GEO-SYNCHRONOUS SATELLITES ATTITUDE DETERMINATION AND CONTROL SUBSYSTEM

In order to understand the function of the fixed momentum wheel, the concept of geo-synchronous satellite attitude determination and control has to be explained briefly. Geo-synchronous satellites are spacecrafts which are put in an equatorial orbit that is synchronous with Earth's rotation around its axis, which means that such satellites have a period of 24 hours per revolution, and hence they always see the same intended coverage area. This particular circular orbit is determined by space mechanics to be 42,000 km away from Earth's center, and it is possible that only three geo-synchronous satellites are sufficient to cover every spot on Earth [2.2].

Because of the unique characteristics of this orbit, and because of the limited space on it, most of the spacecrafts that are put into this orbit are TV broadcasting satellites. The quality of the broadcast TV programs via those satellites is very much related to the attitude of the satellites. The attitude of a spacecraft means its position and its orientation, which is always described using the spacecraft frame of reference shown in Fig. (1.1).

![Fig. (1.1) Geo-synchronous Satellite Frame of Reference [2.2]](image-url)
The change in position or orientation, which a geo-synchronous satellite might encounter, stems from the external torques acting upon the satellite. Some of these external disturbances are gravitational attraction from Earth, Sun, Moon and planets [2,3]. There are two types of orbital position changes, which are drifts towards North and South and drifts towards East and West. There are three types of orientation changes: the first type is rotation around ± X (ROLL) axis, which means a change of the coverage area in the North/South direction with respect to Earth. The second type of orientation changes is rotation of the spacecraft around ± Y (PITCH) axis, which means a change of the coverage area in the East/West direction with respect to Earth, and the third type of orientation changes is rotation of the spacecraft around its ± Z (YAW) axis.

The Attitude Determination and Control Subsystem (ADCS), depicted in Fig. (1.2), is the part of the spacecraft that is responsible for measuring attitude data, processing it, and feeding the control commands to the actuators to correct any change in position or orientation. The ADCS is organized around a central control equipment, and it includes various sensors for the input of attitude data, such as Gyroscopes, Infra Red Earth Sensors and Sun Elevation Sensors, and it also includes the means to change attitude with the Fixed Momentum Wheel (FMW), Thrusters and Solar Arrays [2,4].

![Fig. (1.2) Schematic Diagram for the Attitude Determination and Control Subsystem [2,4]](image-url)
1.3 Momentum Wheels

Momentum wheels are used on various types of spacecrafts as attitude control actuators. They provide angular momentum stabilization and reaction torques. This section provides insight into the function and design of fixed momentum wheels, which are used in geo-synchronous satellites, and proceeds with a brief description of other types of momentum wheels and their conventional usage. Finally a new application of the wheels is presented.

1.3.1 Fixed Momentum Wheels

The function of any fixed momentum wheel in a spacecraft is to continuously provide a constant angular momentum vector in a specific direction. This momentum vector provides gyroscopic stiffness in the plane perpendicular to the momentum vector. For example, if a wheel is mounted on a satellite so that its momentum vector is aligned with the satellite Y (PITCH) axis as shown in Fig. (1.3), the created gyroscopic stiffness prevents the satellite from rotating around any axis in the X-Z plane, and therefore prevents any misalignment of orientation in this plane. If this satellite is a TV broadcasting satellite, and a fine control is maintained over the wheel, it is guaranteed that there is never any disturbance in the coverage area in the North-South direction with respect to Earth. Furthermore, the wheel is capable of accelerating or decelerating under the command of the ADCS so as to provide reaction torques to kill the effect of any disturbing external torques around the Pitch axis. This second function allows the wheel to control the satellite so as to maintain a stable coverage area in the East-West direction with respect to Earth.

The above one-wheel strategy is adopted in many satellite designs since it perfectly controls the coverage area, whereas other degrees of freedom are controlled by thrusters. In other design strategies, three wheels are used to control the three degrees of freedom of the spacecraft orientation whereas thrusters are used to control the position only. It was found by McClamroch et al [1.2] that only two momentum wheels can control the 3 degrees of freedom of the orientation of a spacecraft.
Fig. (1.3) FMW Momentum Vector after Wheel Spin-up at end of Transfer Orbit [2.7]

The fixed momentum wheel under study is mounted on many of the EUROSTAR Platform geo-stationary satellites; it is manufactured by Teldix Co. of Germany. It is an off-the-shelf product that is flight-proven. The FMW under study has a maximum momentum storage capacity of 55 Nms at an angular speed of 5060 rpm. The total mass is 7.5 kg, while the rotor mass is 4 kg. The maximum permissible reaction torque is 0.1 Nm. The lifetime of this product is limited by the angular contact ball bearings unit, and it is about 15 years. A physical presentation of the wheel with the dimensions in (mm) is shown in Fig. (1.4), while a cross sectional view is shown in Fig. (1.5). The complete performance characteristics can be found in Appendix (A) [2.5].
Fig. (1.4) Physical Presentation of a 55 Nms FMW [2.4]

Fig. (1.5) Cross-sectional Presentation of a 55 Nms FMW Design [2.4]

<table>
<thead>
<tr>
<th>Position</th>
<th>Designation</th>
<th>Designation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Base plate</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>Inner Bearing Housing</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>Bearing Unit</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>Outer Bearing Housing</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>Motor Rotor</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>Wheel Hub</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>Spoke</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>Rotating Mass</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>17</td>
</tr>
</tbody>
</table>
The wheel rotor is made out of an aluminum alloy and it is designed so as to maximize the wheel mass moment of inertia around its axis of rotation. The rotor is composed of a hub, 5 1-section inclined spokes and a rotating mass on the outside as shown in Fig. (1.6) [2.6]. The Eurostar model satellites use two Teldix FMW in cold redundancy (standby reserve) on each satellite as shown in Fig. (1.7) [2.7].

Fig. (1.6) Teldix 55 Nms FMW [2.6]

Fig. (1.7) Mounting of Two FMW's on Eurostar Satellites [2.7]
One of the major suppliers of momentum wheels in the American Market is Space Systems Co., an affiliate of Honeywell Inc. The model HM 4520 shown in Fig. (1.8) is the one that gives close performance to the Teldix product. This wheel has a maximum momentum storage capacity of 60.75 Nms at a speed of 5400 rpm. The maximum permissible reaction torque is 0.135 Nm. The mass of the wheel is pretty high (11.1 kg) compared to the Teldix wheel. The rotor is made out of a single piece to ensure stability and balance at high speeds. This design strategy is very cautious about the whirling effect that is encountered at high speeds. The bearings used in the design of this wheel are duplex fixed and floating bearings [2.8]. The detailed characteristics of this wheel are provided in Appendix (B).

![Diagram of Honeywell 60 Nms FMW](image)

Fig. (1.8) Honeywell 60 Nms FMW [2.8]
1.3.2 Other Types of Space Momentum Wheels and Their Usage

Other than the fixed momentum wheel that is described in the previous section, and which this study focuses on, there are two other types of space momentum wheels which are worth mentioning. The first of these categories is Reaction Wheels and the second is magnetic bearing momentum Wheels. It is important to describe those two types because of two reasons: firstly, a special design criterion that is used in the second category is adopted in the proposed design of the FMW, and secondly, the analysis used in the design of the FMW Rotor can be applied to those two types.

Reaction Wheels have the same concept of fixed momentum wheels. Generally, they have the same design, with more or less same operating speeds, lower mass and angular momentum, and smaller sizes. They are mainly used for Low Earth Orbit (LEO) satellites. They satisfy the requirement of cost-effective small satellite designs for both commercial and scientific applications. Generally, three reaction wheels are used in LEO satellites to perform the reorientation maneuvers as needed. This concept is applied in a scientific experiment for a small 60 kg autonomous LEO satellite [2.9]. This concept has also been adopted in the design of LEO multiple-satellite communication networks such as the ORBCOMM constellation [2.6]. The description of both Teldix and Honeywell Reaction wheels are included in Appendix (C).

Magnetic Bearings Momentum Wheels are receiving wide attention in the United States, Europe and Asia because of their valuable advantages over the classic momentum wheels. They basically depend on the idea of using magnetic bearings as a suspension element instead of conventional ball bearings. The use of the magnetic bearings offers many advantages to the design of the wheel, the most important being the elimination of any interaction between the rotor and any other component in the wheel. Consequently, no friction occurs during the spin of the rotor and no lubrication is needed. Akishita [1.3] mentioned other advantages such as: lower energy consumption, longer lifetime and higher reliability. More advantages include extremely low vibrational noise and higher speed capability, which means less weight [2.6].
The more attractive advantage of the Magnetic Bearing Momentum Wheels is the ability of orthotropic attitude control using a single wheel, which means that by using a single flywheel suspended by magnetic bearings, three-axis attitude control and cross momentum storage are achievable. This is done by tilting the rotor using a control circuit connected to the central control processing unit, which obtains the attitude data from the sensors [2.6]. According to Akishita [1.3], there are 5 types of magnetic suspension which can be classified according to the degrees of freedom they control. The more degrees of freedom involved, the more complex the control circuit becomes.

It is important to mention that there are several loading conditions applied on the various types of wheels. The first is the centrifugal effect due to rotation. Other loading conditions include reaction torques or in other words, the stresses resulting from wheel spin-up or spin-down cases. In the case of Magnetic Bearings Momentum Wheels, an additional loading case should be considered, which is the bending stresses resulting from moments created from tilting the rotor to obtain cross momentum storage, or to control the attitude of more than one axis.
1.3.3 Use of Wheels as Energy Storage Systems

A creative idea concerning the use of Magnetic Bearings High Speed Flywheel as an energy storage system for LEO satellites has been developed and implemented in the University of Maryland in cooperation with NASA and Flywheels Systems Inc. The flywheel system is intended to replace batteries that are used during Sun eclipses as the only source of power supply in LEO satellites [2.10]. The function of the batteries is to get fully charged through the main source of spacecraft power, which is solar (photovoltaic) cells during the normal operating conditions, and supply the spacecraft with the power needed during eclipse, while the main source of power is incapable of doing so. Although batteries have been the alternative flight-proven energy source for a long time, there are many problems associated with them, the most important being their weight, especially because they are made from ceramic materials.

The disassembled flywheel system that was developed in the University of Maryland is shown in Fig. (1.9). It consists of five major components: the housing, the magnetic bearings, the motor/generator, the touch down surfaces, and the rotor. This wheel is vacuum sealed and is capable of providing an energy of 50 Wh [2.10].

Fig. (1.9) Disassembled LEO satellite Energy Storage Wheel [2.10]
The motor/generator unit acts as a motor while the wheel is being charged (spin up to 671,000 rpm) by the photovoltaic energy source, and it acts as a generator during eclipse, or in other words, during the discharge 30 min period (spin down to 168,000 rpm). The rotor is composed of two cylinders mounted inside each other. The inner cylinder is made from steel, which enables the rotor to interact correctly with the bearings and the motor/generator unit. The outer cylinder is made from a graphite-reinforced-epoxy, which has high specific strength and stiffness, that help in coping with such high speeds [2.10].

An older design of the same wheel was presented by Kirk [1.4] in 1991. This design provides 500 Wh of energy with a maximum speed of 60,000 rpm. The composite cylinder of the rotor is proposed to be manufactured out of interference assembled rings. According to Kirk [1.4], the usage of interference between the rings allows for application of higher speeds.
1.4 Scope of Work

The main objective of this study is to design a light weight rotor that can be used in conjunction with magnetic bearings for a fixed momentum wheel, which provides a constant angular momentum of 55 Nms. The use of low density materials is evidently a necessity. This criterion entails the application of high speeds, which in turn develop great stress and strain values that are proportional to the squared value of the wheel's angular velocity. This requirement of selecting a material with high specific elastic modulii and strength leads to thinking about composite materials.

The main objective of this study is to come up with a mathematical model, which characterizes the behavior of the fiber-reinforced-rotor under the application of high angular velocities in the domain of the elasticity theory. The anisotropic characteristics of composite laminates make the elastic analysis of such problems quite complex. In this study, closed form mathematical models are developed for the analysis of the behavior of isotropic as well as axi-symmetric (balanced) laminated rotors with interferences, and under the centrifugal effect stemming from the application of constant angular velocity. The mathematical formulation for the case of axi-nonsymmetric (orthotropic) rotor behavior is developed and a numerical solution using the finite element method is presented.

Using these models, several studies are carried out to investigate the appropriateness of each type of material to the design of the application in hand. Firstly, the study compares the cases of using conventional rotor materials such as aluminum alloy and the use of unidirectionally-graphite-reinforced-epoxy laminates. Secondly, the two types of laminate designs are analyzed in the study: the axi-symmetric angle ply laminates and the axi-nonsymmetric orthotropic angle ply laminates. For the case of laminated rotor design, and for the required 55 Nms of angular momentum, a specific rotor mass, and at a specific rotor angular speed, there is a number of parameters which are analyzed and optimized in the study. The parameters under consideration are:

- the amount of interference between neighboring rings,
- the fiber volume fraction in each ring,
- the lamination angle of each ring, and
- the number of rings.

With the required momentum output of 55 Nms, and with the restrictions on the dimensions of the wheel due to the limitations on the mounting space of the wheel inside the spacecraft, one is left with no option but to stick to the same dimensions of the rotor to maximize the rotor inertia and try to optimize other design parameters. The inner radius is 45 mm and the outer radius is 150 mm. The length of the rotor is to be determined based on the target mass of the rotor, which is a function of the material density. The angular velocity is specified by dividing the angular momentum by the rotor mass moment of inertia.

The use of magnetic bearings makes it necessary that the part of the rotor that surrounds the bearings be made of steel. Furthermore, it is suggested by Kirk [1.4] and Jeong [1.5] that if the rotor is made out of concentric rings, the interference in-between the rings created by shrink fits helps in reducing the hoop stresses. Therefore, a design of a rotor that is made out of concentric rings, the innermost ring is made out of steel and has a thickness of 5 mm and the rest of the rotor is made out of a number of concentric rings, which are made from graphite-reinforced-epoxy laminates, is going to be studied.

All the tools required for the analysis of the above mentioned cases are explained and evaluated in the subsequent sections of chapter one.
1.5 Estimation of Composite Materials Properties

In the analysis of the case studies involving composite materials, the properties of the material have to be estimated based on the constituent fibers and matrix properties and also based on the reinforcement volume fraction. The properties of interest here are elastic modulii, ultimate strengths, and coefficient of thermal expansion of the composite materials. There are many models in the literature for the evaluation of the elastic modulii and the coefficients of thermal expansion, however, there are fewer models for the evaluation of the ultimate strengths. Most of the models are presented and analyzed briefly hereafter.

1.5.1 Elastic Modulii

The models used to estimate the overall elastic modulii of a composite material can be classified into two main categories: the mechanics (strength) of materials approach and the theory of elasticity approach. While the first approach uses only the elastic modulii of the fibers and the matrix materials and their respective volume fractions, the second approach adds to these factors principles of the elasticity theory, such as the shape of the reinforcement, their interaction with the matrix, the spacing between the fibers, and the amount of voids in the matrix and around the fibers [1.6].

Mechanics of Materials Approach: Jones [1.6] denoted all the properties of the fiber material by “f” and all the properties of the matrix by “m”. The fibers volume fraction is \( c_f \) and it varies from 0 to 1, and the matrix volume fraction \( c_m = 1 - c_f \). Let \( x_l \) be the fiber direction, using the rule of mixtures, this approach estimates the properties of the composite material in the principal material directions to be [1.6]:

\[
E_{11} = c_f E_f + c_m E_m 
\]
\[
E_{22} = \frac{E_f E_m}{c_m E_f + c_f E_m} = E_{33} 
\]
\[
\nu_{12} = c_m \nu_m + c_f \nu_f = \nu_{13} 
\]
\[
G_{12} = \frac{G_m G_f}{c_m G_f + c_f G_m} = G_{13} 
\]

16
Obviously this methodology has many inaccurate assumptions, for example, it assumes that the transverse stresses in the fibers and the matrix are the same, which is very unlikely. This assumption is used for the prediction of \( E_{33} \) and \( G_{13} \), therefore their values are doubtful.

There are many variations to this model, some of them have tried to find more accurate models, such as the one by Ekval [1.7], which presented a model that accounts for the tri-axial stress state in the matrix due to fiber resistance. Ekval’s estimated expressions are:

\[
E_{11} = c_f E_f + c_mE_m' \tag{1.5}
\]

\[
E_{22} = \frac{E_f E_m'}{c_f E_f + c_mE_f (1 - v_m^2)} = E_{33} \tag{1.6}
\]

\[
E_m' = \frac{E_m}{1 - 2v_m^2} \tag{1.7}
\]

Although this model is more effective than the previous simple one, it does not work well if \( v_m < 0.25 \) [1.7].

**Elasticity Approach:** There are many elasticity models for the estimation of composite material elastic moduli; the most famous approaches are the variational techniques using energy bounding principles, solutions with contiguity, and exact solutions [1.6].

1. **Bounding Techniques:** those techniques depend mainly on the linear elasticity energy principles to predict upper and lower limits on different moduli. The work started out by Paul [1.8] concentrated on alloyed isotropic homogenous material, and it was then reinterpreted to be used for heterogeneous composite materials by Hashin and Shtrickman [1.9]. Their model treated the heterogeneous material as an elastic sphere inside a concentric spherical portion of elastic matrix material. The major inaccuracies of this model stem from the assumption that the spherical inclusions never touch each others, which is not true on the practical level, especially when the reinforcement volume fraction increases. Hashin and Rosen [1.10] extended this work to a new model that encompasses fiber-reinforced-materials. The fibers are assumed to have a circular hollow or solid cross section distributed in a hexagonal or random arrays. The problem with this
model is that the arrays used are not typical distributions in the practical fiber-reinforced materials.

2- Elasticity Solutions with Contiguity: the idea behind these solutions is based on the fact that because the fibers inside the matrix are distributed randomly, especially the graphite/epoxy combination, there is no guarantee that the fibers touch or do not touch each other. Hence, the solution must incorporate this fact by using a linear combinatorial solution which adds up two solutions representing the two situations when the fibers touch or do not touch each other. The problem associated with those types of models is that their results must be compared to experimental values to determine the amount of contiguity between the fibers [1.6]. Chamis [1.11] presented a simple model representing the contiguity principles and includes the voids' effect. The model is illustrated in Fig. (1.10), and the modulii estimations are presented in equations (1.8) through (1.21). It is obvious that it is difficult to predict the required input data of the model; it will need extensive experimental work.

![Diagram](image.png)

Fig. (1.10) Chamis’ Model for Prediction of Lamina Stiffness Modulii
Partial Volumes: $c_f + c_m + c_o = 1$ \hspace{1cm} (1.8)

Ply Density: $\rho = c_f \rho_f = c_m \rho_m$ \hspace{1cm} (1.9)

Resin Volume Ratio: $c_m = (1 - c_o)/(1 + (\rho_m/\rho_f)(1 - c_o))$ \hspace{1cm} (1.10)

Fiber Volume Ratio: $c_f = (1 - c_o)/(1 + (\rho_f/\rho_m)(1 - c_f))$ \hspace{1cm} (1.11)

Weight Ratios: $\lambda_f + \lambda_m = 1$ \hspace{1cm} (1.12)

Ply Thickness: $t = 0.5(N_f/d_f \sqrt{\pi/c_f})$ \hspace{1cm} (1.13)

Interply Thickness: $\delta_i = 0.5(\sqrt{\pi/c_f} - 2) \ d_f$ \hspace{1cm} (1.14)

Interfiber Spacing: $\delta_s = \delta_i$ \hspace{1cm} (1.15)

Longitudinal Modulus: $E_{11} = c_f E_{11} + c_m E_m$ \hspace{1cm} (1.16)

Transverse Modulus: $E_{22} = \frac{E_m}{1 - c_f (1 - E_m/E_{22})} = E_{33}$ \hspace{1cm} (1.17)

Shear Modulus: $G_{23} = \frac{G_m}{1 - c_f (1 - G_m/G_{23})} = G_{12}$ \hspace{1cm} (1.18)

Shear Modulus: $G_{35} = \frac{G_m}{1 - c_f (1 - G_m/G_{35})}$ \hspace{1cm} (1.19)

Poisson’s Ratio: $\nu_{12} = c_f \nu_{12} + c_m \nu_m = \nu_{23}$ \hspace{1cm} (1.20)

Poisson’s Ratio: $\nu_{13} = c_f \nu_{13} + c_m (2\nu_m - \nu_{12} E_{11}/E_{22})$ \hspace{1cm} (1.21)

3- Exact Solutions Techniques: The idea of exact solutions is to first assume part of the solution and then try to satisfy the equilibrium equations and the boundary conditions [1.6]. One of the first methods of this approach is the Saint Venant semi-inverse method [1.6], whose one of its assumptions is that plane sections remain plane. The exact solution methods depend to a great extent on the geometry and characteristics of the fibers used in the analysis. It also depends on the types of arrays considered in the analysis. It has been found that a hexagonal array representation matches closely the experimental results. This is justified on the basis that this type of distribution has more randomness than other shapes of arrays. Two types of variations from this method brought about good results.
The two models are: the Self-Consistent model and the Mori-Tanaka model. The advantage of these two models is that they don’t specify a certain type of arrays [1.6]. The Self-Consistent Model has many versions, the one by Whitney and Riley [1.12] has a single hollow circular fiber embedded in a concentric cylinder of matrix material. On the other hand, Mori-Tanaka model [1.13] considers a single elliptical cross-section fiber in a matrix material medium. The advantage of the Mori Tanaka model over the Self-Consistent model is that the first gives direct method for estimating the stiffness matrix Hill’s modulii, whereas the second model has a more lengthy solution for obtaining those modulii [1.14]. Consequently, the Mori Tanaka model is used in this study for the evaluation of the unidirectionally fiber-reinforced-laminae stiffness matrices, the model is presented in section (2.1)
1.5.2 Strength

The models that were developed for the estimation of the ultimate strengths of composite materials are not as many as those developed for the estimation of stiffness modulii. Most of those models are based on the mechanics of materials approach. In general, it is advised to perform standard tests on any new composite material to measure its ultimate strengths. However, there are some models of interest which correlate more or less well with experimental results [1.6]. Weeton [1.15] presented a simple model used for the estimation of ultimate tensile strength in the fiber direction, which is mainly based on the rule of mixture as seen in the following equation:

\[ S_{11T\text{ (composite)}} = S_{T} \varepsilon_{T} + S_{m}^{*} \varepsilon_{m} \]  

(1.22)

Where \( S_{m}^{*} \) is the tensile strength of the matrix at a matrix strain \( \varepsilon_{m} \) equal to the ultimate tensile fiber strain \( \varepsilon_{T} \), occurring at fiber tensile strength \( S_{T} \) [1.15]. Although this model is effective in the calculation of the ultimate tensile strength of the composite material in the fiber direction, it does not have estimates for the other ultimate strengths of the composite material. Furthermore, it is difficult to find \( S_{m}^{*} \) from materials tables because only ultimate strengths are provided into such tables and not the complete tension test plots.

Another major inaccuracy of the above presented model is that it is build on the assumption that all the fibers fail together, which is not realistic due to the non-conformances of the manufacturing techniques. Therefore, it is advised by many experts to use statistical strength distribution models to have more accurate estimation of the composite ultimate strength [1.6].

Another more comprehensive model which estimates the five in-plane uniaxial strengths: longitudinal tension \( S_{11T} \), longitudinal compression \( S_{11C} \), transverse tension \( S_{33T} \), transverse compression \( S_{33C} \), and inter-laminar shear \( S_{13S} \), is presented by Chamis [1.11]. This model assumes that the matrix is a ductile material while the reinforcement material is of a more brittle nature. This model is used in the analysis and is presented in section (2.2).
1.5.3 Coefficient of Thermal Expansion

The models used for predicting the composite material coefficients of thermal expansion (CTE) do not only depend on the thermal coefficients of expansion of the fiber and matrix materials and their respective volume fractions, they also use the matrix, fibers, and composite elastic moduli. Most of the models developed for the estimation of the lamina elastic moduli are used in the estimation of lamina CTE’s [1.6]. The following are the solutions obtained by a simple model developed by Chamis [1.11], for calculating the composite material CTE’s in the longitudinal and transverse directions. Although this model is effective, it does not calculate the CTE’s in all directions [1.11].

Longitudinal CTE: \[ \alpha_{11} = \frac{\epsilon_f \alpha_{f11} E_{f11} + \epsilon_m \alpha_m E_m}{E_{11}} \] (1.23)

Transverse CTE: \[ \alpha_{22} = \alpha_f \frac{\sqrt{\epsilon_f} + (1 - \sqrt{\epsilon_f})(1 + \epsilon_f \nu_m E_{f11}/E_{11})}{\alpha_m} = \alpha_{33} \] (1.24)

Another method based on the Mori-Tanaka model for estimating composite materials stiffness moduli, is used in the analysis and is developed in section (2.3).

The special thermal characteristic of the graphite fibers, that they have a negative coefficient of thermal expansion in the longitudinal direction makes them very attractive for use in aerospace structures. If a structure is made from graphite-reinforced-balanced (±φ) laminates, there exist two angles, at which the overall longitudinal CTE is equal to null, which means that, at those angles, no matter how much increase or decrease in temperature the structure faces, it will not deform in the fibers direction [1.16]. This idea can be used in preparing curves of CTE versus lamination angle variation for several fiber volume fractions, which can be useful in calculating the amount of allowable interferences between the momentum wheel concentric rings.
1.6 Estimating the Overall Properties of Composite Laminates

After evaluating the properties of a fiber-reinforced-lamina in the principal material directions 1, 2 and 3, it is needed to evaluate the properties in a reference frame of interest, which in this type of analysis, is the polar coordinates in which the loading is described. The global frame of reference has the axes $\theta$, $z$, and $r$, and the loading of the centrifugal force is in the radial direction. In this study, $z$ coincides with 2, and the rotation angle $\phi$ is a positive rotation around 2 or $z$ from $\theta$ to 1, or from $r$ to 3 as shown in Fig. (1.11).

![Diagram showing global and local coordinate systems for a uni-directionally reinforced lamina.]

Fig. (1.11) Global and Local Coordinate Systems for Uni-directionally Reinforced Lamina

By taking the direction cosines, the local stress and strain tensors of a lamina in the 1-2-3 coordinates can be transferred to the global stress and strain tensors of the lamina in the $\theta$-$z$-$r$ coordinates by the following equations [1.6]:

$$
\begin{bmatrix}
\sigma_{\theta} \\
\sigma_z \\
\sigma_r \\
\sigma_{\varphi} \\
\sigma_{\theta r}
\end{bmatrix} =
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{13} \\
\tau_{12}
\end{bmatrix} \cdot [T]^{-1}
$$

(1.25)
\[
\begin{bmatrix}
\varepsilon_0 \\
\varepsilon_z \\
\varepsilon_r \\
\frac{\gamma_{xz}}{2} \\
\frac{\gamma_{xz}}{2} \\
\frac{\gamma_{rr}}{2} \\
\frac{\gamma_{zz}}{2}
\end{bmatrix}
= [T]^t
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{13} \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{32}
\end{bmatrix}
\]

(1.26)

where \( T \) denotes the transformation matrix shown below:

\[
[T] =
\begin{bmatrix}
\tau^2_{11} & \tau^2_{12} & \tau^2_{13} & 2\tau_{12}\tau_{13} & 2\tau_{11}\tau_{13} & 2\tau_{11}\tau_{12} \\
\tau^2_{21} & \tau^2_{22} & \tau^2_{23} & 2\tau_{22}\tau_{23} & 2\tau_{21}\tau_{23} & 2\tau_{21}\tau_{22} \\
\tau^2_{31} & \tau^2_{32} & \tau^2_{33} & 2\tau_{32}\tau_{33} & 2\tau_{31}\tau_{33} & 2\tau_{31}\tau_{32} \\
\tau_{21}\tau_{31} & \tau_{22}\tau_{32} & \tau_{23}\tau_{33} + \tau_{13}\tau_{13} & \tau_{22}\tau_{32} + \tau_{12}\tau_{13} & \tau_{21}\tau_{31} + \tau_{11}\tau_{13} & \tau_{21}\tau_{31} + \tau_{11}\tau_{12} \\
\tau_{11}\tau_{31} & \tau_{12}\tau_{32} & \tau_{13}\tau_{33} + \tau_{13}\tau_{12} & \tau_{12}\tau_{32} + \tau_{11}\tau_{13} & \tau_{11}\tau_{31} + \tau_{11}\tau_{13} & \tau_{11}\tau_{31} + \tau_{11}\tau_{12} \\
\tau_{11}\tau_{21} & \tau_{12}\tau_{22} & \tau_{13}\tau_{23} + \tau_{13}\tau_{22} & \tau_{12}\tau_{22} + \tau_{11}\tau_{23} & \tau_{11}\tau_{21} + \tau_{11}\tau_{23} & \tau_{11}\tau_{21} + \tau_{11}\tau_{22}
\end{bmatrix}
\]

(1.27)

\( t_{ij}, \ i, j = 1, 2, 3, \) are direction cosines of the 1,2,3 local frame of reference with respect to the \( \theta, z, r \) global frame of reference [1.6]. If a positive rotation of angle \( \phi \) around \( z \) or \( 2 \) is encountered, then the direction cosines can be defined as:

\( t_{11} = \cos \phi, t_{12} = 0, t_{13} = \sin \phi, t_{21} = 0, t_{22} = 1, t_{23} = 0, t_{31} = -\sin \phi, t_{32} = 0, \) and \( t_{33} = \cos \phi. \)

From equations (1.25) and (1.26), and the stress-strain relationships described in Appendix (D), the global compliance and the stiffness matrices of the lamina can be obtained as follows [1.6]:

\[
[M]_{global} = [T]^t [M]_{local} [T]
\]

(1.28)

\[
[L]_{global} = [M]_{global}^{-1}
\]

(1.29)

It should be noticed that if the fibers are transversely isotropic, such as the case with the graphite fibers, and the matrix is fully isotropic, such as the epoxy resin, the transversely isotropic lamina local stiffness matrix takes the form shown in equation (1.30).
\[
[L]_{\text{local}} = \begin{bmatrix}
  n & l & l & 0 & 0 & 0 \\
  l & k+m & k-m & 0 & 0 & 0 \\
  l & k-m & k+m & 0 & 0 & 0 \\
  0 & 0 & 0 & m & 0 & 0 \\
  0 & 0 & 0 & 0 & p & 0 \\
  0 & 0 & 0 & 0 & 0 & p \\
\end{bmatrix} \quad (1.30)
\]

where \( k, l, m, n \) and \( p \) are lamina Hill’s modulii, which can be estimated using the methods described in section (1.5.1).

The transformed global stiffness matrix takes the form of:

\[
[L]_{\text{global}} = \begin{bmatrix}
  L_{11} & L_{12} & L_{13} & 0 & L_{15} & 0 \\
  L_{21} & L_{22} & L_{23} & 0 & L_{25} & 0 \\
  L_{31} & L_{32} & L_{33} & 0 & L_{35} & 0 \\
  0 & 0 & 0 & L_{44} & 0 & L_{45} \\
  L_{51} & L_{52} & L_{53} & 0 & L_{55} & 0 \\
  0 & 0 & 0 & L_{64} & 0 & L_{65} \\
\end{bmatrix} \quad (1.31)
\]

where \( L_{12} = L_{21}, L_{13} = L_{31}, L_{23} = L_{32}, L_{15} = L_{51}, L_{25} = L_{52}, L_{35} = L_{53} \) and \( L_{46} = L_{64} \).

Note that \( L_{15} \) and \( L_{25} \) are now not equal to zero; those are called by Lekhnitski [1.17] the coefficients of mutual influence. The name is indicative of the effect of these stiffness factors; they create a coupling effect between extension and shearing deformation. Therefore, under plane stress conditions, if a pure normal load is applied in the \( \theta \) or \( r \) directions, a shearing (twisting) effect is induced in the \( \theta r \) plane. The opposite effect is also encountered; i.e., when a pure twisting couple is applied in the \( \theta r \) plane, normal stresses and hence extensions are induced in the \( \theta \) and \( r \) directions [1.17].

It is also important to state that this transformation is used to calculate the local stresses from the global ones. This step is used to compare local stresses to lamina’s strengths explained in section (1.5.2) using a failure criterion explained in section (1.7).

Local thermal coefficients of expansion of a lamina need also be transformed in global coordinates so that one can calculate the amount of change in temperature needed to be applied to the lamina to expand it with the amount of interference required in the global coordinates.
"A laminate is two or more laminae bonded together to act as an integral structural element" [1.6]. The estimation of the global properties of a laminate is based on the global properties of the constituent laminae and their respective thicknesses. In the model developed under the umbrella of the Classical Lamination Theory, and if it is assumed that each lamina in a laminate has the same type and volume fraction of fibers, the only variation is the ply angle (θ). The laminate stiffness matrix is derived from the constituent laminae stiffness matrices. The laminate thermal stiffness matrix is also calculated using the constituent laminae stiffness and thermal stiffness matrices [1.6].
1.7 Failure Criteria

There are many criteria for the prediction of failure of isotropic materials. The simplest is to use the yield strength of the ductile material as an upper limit on the stresses induced in the body due to loading. However for the case of composite structures, the criteria are not that straightforward. The failure mode of a composite material is of various natures. It can be due to “fiber breakage, loss of fiber stability under compression, matrix fracture, stack delamination, weakening of fiber-matrix interface adhesion leading to formation and growth of cracks, etc” [1.18]. It is impossible to present a model that accounts for all the above failure modes.

The existing criteria can be divided into two groups; the first one does not account at all for the interaction between the different failure modes. The maximum stress and maximum strain theories are among such criteria [1.18]. The idea of the maximum stress theory is that the stresses in the principal material directions must be less than the ultimate strengths of the lamina in those directions, whereas the maximum strain theory limits the strains in the lamina principal material directions to the ultimate strains along these directions. Both theories do not account for the possible various failure modes, and both theories are not affected by the sign of the shear stress [1.6].

The second type of failure criteria is one in which the interaction between failure modes are accounted for with variable degrees. Some of those models are Tsai-Hill theory and Tsai-Wu tensor theory [1.6]. The Tsai-Hill [1.19] theory is an extension to the isotropic material von Mises yield criterion, which is concerned with the amount of energy used to distort the body rather than the change of its volume. The advantage of this criterion is that it has a single equation for the determination of the failure status, which is not the case for the maximum stress or maximum strain theories, where there are 3 criteria to be satisfied simultaneously. Tsai-Hill theory’s results agree to a great extent with experimental results for some materials. This theory is also plausible because it accounts for considerable interaction between failure modes. The disadvantage of this theory is that for a general orthotropic lamina under biaxial stress conditions, distortion never occurs without extension and vice versa [1.6].
Tsai-Wu [1,20] tensor theory is a good methodology for the estimation of failure of a composite lamina under plane stress conditions. Its major advantage over the other theories is that it includes more terms in the prediction equation, which makes it correlate better with experimental results. It has more curve fitting ability and includes various material strengths in tensor form. This theory is used in the analysis and is presented in section (2.4).
1.8 Elastic Analysis of Axi-symmetric and Axi-nonsymmetric Laminated Concentric Rings with Interferences Under Plane Stress Conditions

The elastic analysis of fully isotropic concentric rings with interferences under plane stress conditions is a mature subject in the theory of elasticity. In fact, this axi-symmetric plane polar coordinate problem is always presented in the theory of elasticity literature as a special type of problems whose closed-form solution in terms of stresses, strains and displacements, is easily obtainable. The famous application for this type of problems is the two concentric cylinders used for gun barrels, which are used to reduce the high hoop stresses which result from firing operations; however, this is a plane strain problem.

The analysis of such a system becomes more difficult when the rings are made from composite materials due to the anisotropy of the material. Despite this fact, the analysis of such a case is highly needed because of the tailored properties the composite materials provide for the system under consideration. There are two types of systems to be analyzed: the first is axi-symmetric angle ply laminates and the second is orthotropic (axi-nonsymmetric) angle ply laminates.

Symmetric angle ply laminates have a distinct characteristic, that they do not exhibit coupling between extension and shearing. In other words, the terms $L_{\text{global}15}$, $L_{\text{global}25}$, $L_{\text{global}51}$ and $L_{\text{global}52}$ have a null value. Axi-symmetric laminates can be divided into three categories according to their lamination angle. The first type is $0^\circ$ angle laminates, the second is $90^\circ$ angle laminates, and the third is balanced angle ply laminates. Balanced angle ply laminates must be composed of an even number of laminae which are distributed in a mirror-like structure with respect to the laminate middle plane of symmetry. For example, the laminae orientation stack $(+20^\circ, -20^\circ, +20^\circ, -20^\circ || -20^\circ, +20^\circ, -20^\circ, +20^\circ)$ is a balanced symmetric laminate, that is denoted as $(\pm 20^\circ)_{2s}$ balanced laminate. The symbol $||$ denotes the plane of symmetry. The three types of axi-symmetric laminates are depicted in Fig. 1.12

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(a) $0^\circ$ Angle Laminates  (b) $90^\circ$ Angle Laminates  (c) Balanced Angle Ply Laminates

Fig. (1.12) Types of Axi-symmetric Laminates (Exploded Views)

Under plane stress conditions, the global stiffness matrix can be reduced to $3 \times 3$; the stiffness matrix of a balanced laminate takes the following form:

$$
[L]_{global} = \begin{bmatrix}
S & T & 0 \\
T & Z & 0 \\
0 & 0 & H
\end{bmatrix}
$$

(1.32)

where $S = L_{global11}$, $T = L_{global12} = L_{global31}$, $Z = L_{global33}$, and $H = L_{global55}$.

Notice that the diagonal terms are not equal as in the isotropic case. For balanced angle ply laminates, it should be noticed that the terms $L_{global15}$, $L_{global25}$, $L_{global35}$ and $L_{global52}$ vanish although each lamina has those terms not equal to zero. This is because for a balanced laminate:

$$
L_{global15} \text{ of } +\phi \text{ lamina } = - L_{global15} \text{ of } -\phi \text{ lamina}
$$

which is the same for $L_{global25}$, $L_{global35}$ and $L_{global52}$. Therefore, the coupling between extension and shearing effects vanishes, which is shown in Fig. (1.13) [1.6].
Fig. (1.13) Shear Effect on Balanced Angle Ply Laminates

The analysis of concentric rings with interferences of balanced laminates is very limited in the literature. The famous case in which this analysis was applied, is that of the rotor of an energy storage flywheel for LEO satellites developed in the University of Maryland. One of the sources [1.5] that covered this topic presented a model for a ten concentric rings rotor made out of balanced laminates. The analysis utilizes plane strain conditions due to the geometry of the rotor, which has a large length in comparison to its diameter. The reinforcement is oriented in the hoop-axial plane. The authors have used graphite/epoxy material and have optimized two design parameters, namely, the lamination angle of each ring, and the amount of interference at each interface. It was found out that by having zero interferences, and by changing the ply angle from hoop direction to the scenario shown in Fig. (1.14), the angular velocity could be raised from 55,500 to 63,000 rpm. It was also found out that by keeping the fibers oriented along the hoop direction, and use of interferences between the rings as shown in Fig. (1.15), the angular velocity could be raised from 55,500 to 86,900 rpm. Therefore, it is concluded that the amount of interference is of more importance than the lamination angle [1.5].
The second type of analysis is that of orthotropic (axi-nonsymmetric, unbalanced) laminates, which could take many shapes. This type of laminates can have many laminae with a single fiber orientation, that is neither in the hoop nor the radial directions. It can also be a non-symmetric $\pm \phi$ laminate, such that the laminae are not symmetric around the laminate middle plane, for example, the laminae orientation stacks $(+20^\circ, -20^\circ, +20^\circ, -20^\circ \parallel +20^\circ, -20^\circ, +20^\circ, -20^\circ)$ and $(+20^\circ, -20^\circ, +20^\circ, -20^\circ, -20^\circ, +20^\circ, -20^\circ, +20^\circ, -20^\circ)$ are non-symmetric angle ply laminates. The common feature of all those types of orthotropic laminates is the fact that the terms $L_{\text{global}15}$, $L_{\text{global}25}$, $L_{\text{global}51}$ and $L_{\text{global}52}$ are not equal to zero, which creates coupling between the extension and twisting effects. This leads to a very complex formulation for the solution. There are no such closed-form solutions found in the literature.
Chapter Two

Analysis and Modeling

The suggested design for the magnetically-suspended momentum wheel rotor is to be made out of a number of concentric rings with interferences as shown in Fig. (2.1). The innermost ring is made of steel so as to provide the required interaction with the magnetic bearings. Therefore the analysis of isotropic concentric rings is presented. The rest of the rings are either made of aluminum or graphite-reinforced-epoxy laminates. The two alternatives are evaluated in chapter three using the analysis laid in this chapter.

Fig. (2.1) Top and Front Views of the Laminated Multi-Ring Rotor
The dimensions of the rotor are the same as the Teldix 55 Nms FMW. The inner and outer radii are 45 and 150 mm respectively. The steel ring thickness is 5 mm and the thicknesses of the rest of the rings, be it laminated GRE or aluminum rings, are equal and are determined based on the number of the rings. With those fixed radii, and based on the density of the material used, the depth of the rotor is determined for a specific mass. It usually ranges from 9 to 15 mm. For an angular momentum of 55 Nms, and for a specified total rotor mass, the mass moment of inertia is calculated and, hence, the rotor angular speed is determined.

Laminate design can take many shapes. The ones considered in the analysis are: balanced angle ply laminates and general orthotropic laminates. The tools used in the analysis include the calculation of the laminates elastic modulii, strengths, thermal properties, and finally the preparation of the failure criterion.
2.1 Estimation of Elastic Moduli of a Laminated Material in Polar Coordinates

The estimation of the stiffness matrix of a laminated material in a general polar coordinate system proceeds in three steps. First, the Mori-Tanaka model is used for estimation of the lamina properties from its constituent materials' properties and their respective volume fractions. Second, the lamina stiffness matrix is constructed in the global coordinate system by the transformation of the estimated Mori Tanaka local stiffness matrix. Finally, the overall laminate stiffness matrix is obtained by combining the constituent laminae global properties using Classical Lamination Theory (CLT).

The first step in obtaining the lamina stiffness modulii is to get its constituent materials stiffness matrices. The fibers and the matrix stiffness matrices are constructed from Hill’s modulii in the local coordinate systems (principal material directions) as follows [1.23]:

\[
\begin{bmatrix}
    n_s & l_z & l_z \\
    l_z & (k_z + m_z) & (k_z - m_z) \\
    l_z & (k_z - m_z) & (k_z + m_z) \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

where \(f\) and \(m\) denote fibers and matrix Hill’s modulii. For the Mori-Tanaka model, the matrix is assumed to be isotropic, and the fibers are either isotropic or transversely-isotropic [1.13]. In this study, the matrix is epoxy, which is an isotropic material and the P-100 graphite fibers are transversely-isotropic. The resin Young’s modulus \(E_m\) is 2.5 GPa and its Poisson’s ratio \(\nu_m\) is 0.4 [1.23]. The resin shear and Bulk’s modulii \(G_m\) and \(K_m\) can be obtained as follows [1.24]:

\[
G_m = \frac{E_m}{2(1 + \nu_m)}
\]

\[
K_m = \frac{E_m}{3(1 - 2\nu_m)}
\]
The isotropic epoxy matrix Hill's modulii can be obtained as follows [1.23]:

\[ k_m = K_m + \frac{G_m}{3} \]  
\[ l_m = K_m - \frac{2}{3} G_m \]  
\[ n_m = K_m + \frac{4}{3} G_m \]  
\[ m_m = p_m = G_m \]  

(2.2)  
(2.3)  
(2.4)  
(2.5)

The elastic modulii of the transversely-isotropic P-100 graphite fibers are defined in two planes: the axial plane, which is the plane that contains the fibers, and the transverse plane, which is the plane perpendicular to fibers' direction. The axial Young's modulus \( E_{f_a} \) is 690 GPa; The transverse Young's modulus \( E_{f_t} \) is 6.07 GPa; the axial plane Poisson's ratio \( \nu_{f_a} \) is 0.2; the transverse plane Poisson's ratio \( \nu_{f_t} \) is 0.41; and the axial plane shear modulus \( G_{f_a} \) is 15.5 GPa [1.16]. To construct the compliance matrix of the graphite fibers, one more modulus is needed, that is the transverse plane shear modulus, which can be obtained as: \( G_{f_t} = \frac{E_{f_t}}{2(1+\nu_{f_t})} \).

The compliance matrix of the P-100 graphite fibers is [1.14]:

\[
\begin{bmatrix}
1 & \nu_{f_t} & -\nu_{f_t} & 0 & 0 & 0 \\
\nu_{f_t} & E_{f_a} & E_{f_a} & 0 & 0 & 0 \\
-\nu_{f_t} & E_{f_a} & E_{f_a} & 0 & 0 & 0 \\
\nu_{f_t} & 1 & -\nu_{f_t} & E_{f_a} & E_{f_a} & 0 & 0 & 0 \\
-\nu_{f_t} & \nu_{f_t} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{f_t} & 0 \\
0 & 0 & 0 & 0 & -1/G_{f_t} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{f_t} \\
\end{bmatrix}
\]  

(2.6)

and \( L_f = [M_f]^{-1} \)  

(2.7)
The Hill’s modulii for the P-100 graphite fibers can be obtained as follows [1.23]:

\[ k_f = L_{f22} - L_{f44} \]  
\[ l_f = L_{f12} \]  
\[ n_f = L_{f11} \]  
\[ m_f = L_{f44} \]  
\[ p_f = L_{f55} \]  

The Hill’s modulii of the fibers and matrix denoted above are now used in calculating the Mori-Tanaka estimates for the overall lamina local stiffness matrix. Let \( c_f \) denotes fibers volume fraction and \( c_m \) denotes matrix volume fraction, then [1.13]

\[ p = \frac{2c_fP_mP_f + c_m(P_mP_f + P_f)}{2c_fP_m + c_m(P_f + P_m)} \]  
\[ m = \frac{m_m m_f (k_m + 2m_m) + k_m m_f (c_f m_f + c_m m_m)}{k_m m_m + (k_m + 2m_m)(c_f m_f + c_m m_m)} \]  
\[ k = \frac{c_f k_f (k_f + m_m) + c_m k_m (k_f + m_f)}{c_f (k_f + m_f) + c_m (k_f + m_m)} \]  
\[ l = \frac{c_f l_f (k_f + m_m) + c_m l_m (k_f + m_m)}{c_f (k_m + m_m) + c_m (k_f + m_m)} \]  
\[ n = c_m n_m + c_f n_f + (l - c_f l_f - c_m l_m) \frac{l - l_m}{k_f - k_m} \]  

The uni-directionally reinforced lamina overall local stiffness matrix can now be constructed as follows:

\[ [L]_{local} = \begin{pmatrix} 
 n & l & l & 0 & 0 & 0 \\
 l & k + m & k - m & 0 & 0 & 0 \\
 l & k - m & k + m & 0 & 0 & 0 \\
 0 & 0 & 0 & m & 0 & 0 \\
 0 & 0 & 0 & 0 & p & 0 \\
 0 & 0 & 0 & 0 & 0 & p 
\end{pmatrix} \]  

and \[ [M]_{local} = [L]_{local}^{-1} \]
The second step is to get the global stiffness matrix of the lamina in the global loading coordinate system $\theta$, $z$ and $r$ by transforming the local compliance matrix described in the principal material’s coordinate system 1, 2, and 3 as described in section (1.6).

The lamina global stiffness matrix can be obtained by inverting the compliance matrix as follows:

$$[L_{\text{global}}] = [M_{\text{global}}]^{-1}$$ \hfill (2.20)

Now that the global stiffness and compliance matrices are obtained for individual laminae, the global matrices for a laminate that is composed of a number of ($N$) laminae should be obtained. The Classical Lamination Theory provides a simple straightforward technique. The technique is applied here under the condition that each lamina in a laminate has the same type of fibers, and the only variation is in the ply angle ($\phi$) and reinforcement volume fraction $v_f$. If the lamina thickness is denoted by $t_i$ and the total laminate thickness is denoted by $t$, then the global laminate stiffness matrix $[L_{\text{global}}]$ can be calculated as [1.6]:

$$[L_{\text{global}}] = \frac{1}{t} \sum t_i \ [L_{i,\text{global}}]$$ \hfill (2.21)

If all the laminae have the same thickness, then the above equation becomes:

$$[L_{\text{global}}] = \frac{1}{N} \sum [L_{i,\text{global}}]$$ \hfill (2.22)
2.2 Estimation of Strength of a Laminated Material in Principal Material Directions

Chamis [1.11] has presented a comprehensive model which estimates the five in-plane uniaxial strengths: longitudinal tension $S_{11F}$, longitudinal compression $S_{11C}$, transverse tension $S_{33T}$, transverse compression $S_{33C}$, and inter-laminar shear $S_{13S}$ of a lamina in principal material directions (1, 2 and 3) as follows:

1. Longitudinal Tension: $S_{11F} \approx c_f S_{FP}$ \hspace{1cm} (2.23)

2. Longitudinal Compression:
   - Fiber Compression: $S_{11C} \approx c_f S_{FC}$ \hspace{1cm} (2.24)
   - Delamination/Shear: $S_{11C} \approx 10S_{13S} + 2.5S_{mF}$ \hspace{1cm} (2.25)
   - Microbuckling: $S_{11C} \approx \frac{G_m}{1-c_f \left(1 - \frac{G_m}{G_{f13}}\right)}$ \hspace{1cm} (2.26)

3. Transverse Tension: $S_{33T} \approx \left(1 - \left(c_f - c_t\right) \left(1 - \frac{E_m}{E_{f33}}\right)\right) S_{mT}$ \hspace{1cm} (2.27)

4. Transverse Compression: $S_{33C} \approx \left(1 - \left(c_f - c_t\right) \left(1 - \frac{E_m}{E_{f33}}\right)\right) S_{mC}$ \hspace{1cm} (2.28)

5. Interlaminar Shear: $S_{13S} \approx \left(1 - \left(c_f - c_t\right) \left(1 - \frac{G_m}{G_{f13}}\right)\right) S_{ms}$ \hspace{1cm} (2.29)

The terms described in equations from (2.23) to (2.29) are depicted in Fig. (2.2)

![Fig. (2.2) In-plane Lamina Uniaxial Strengths](image-url)
The approximate Epoxy experimental strengths are [1.25]:

\[ S_{mt} = 140 \text{ MPa} \]

It is assumed that:

\[ S_{mc} = 140 \text{ MPa} \]
\[ S_{mS} = 140 \text{ MPa} \]

The approximate P-100 experimental strengths are [1.21]:

\[ S_{TR} = 2.2 \text{ GPa} \]

It is assumed that:

\[ S_{TC} = 2.2 \text{ GPa} \]

It is important to notice that there are three different modes for the longitudinal compression:

1. fiber compression (shear plane) fracture,
2. delamination transverse splitting or panel buckling, and
3. fiber micro-buckling.

A conservative approach is to take the lowest value among the above three longitudinal compressive failure modes. Another less conservative one is to use the average of the two lower values [1.11]. The first approach is going to be used in the analysis.
2.3 Estimation of Thermal Properties of a Laminated Material in Polar Coordinates

The steps needed to estimate the coefficients of thermal expansion of a laminated composite material are the same three steps of estimating the composite elastic moduli. The first step is to get the fibers and the matrix thermal vectors as follows:

\[
\begin{align*}
\mathbf{m}_m &= \begin{bmatrix} \alpha_m & 0 & 0 & \alpha_m \end{bmatrix}^T \\
\mathbf{m}_f &= \begin{bmatrix} \alpha_f & 0 & 0 & \alpha_f \end{bmatrix}^T
\end{align*}
\] (2.30)

where:

For epoxy: \( \alpha_m = 5.6 \times 10^{-3} \, ^\circ\text{C} \) and for P-100 graphite: \( \alpha_f = -1.62 \times 10^{-4} \, ^\circ\text{C} \) and \( \alpha_f = 10.8 \times 10^{-6} \, ^\circ\text{C} \). [1.16]

The lamina thermal strain vector in local coordinates is given by Levin’s equation [1.23] as follows:

\[
\mathbf{m}_{local} = c_m \begin{bmatrix} B_m \end{bmatrix} \mathbf{m}_m + c_f \begin{bmatrix} B_f \end{bmatrix} \mathbf{m}_f
\] (2.32)

where \( c_m \) is matrix volume fraction, \( c_f \) is fibers matrix fraction and \( [B_m] \) and \( [B_f] \) are strain concentration factors and they can be obtained using Mori-Tanaka [1.13] as follows:

\[
\begin{align*}
[B_m] &= \frac{1}{c_m} (\begin{bmatrix} M_n \end{bmatrix} - \begin{bmatrix} M_f \end{bmatrix})^T (\begin{bmatrix} M_n \end{bmatrix} - \begin{bmatrix} M_f \end{bmatrix})^T \\
[B_f] &= \frac{1}{c_f} (\begin{bmatrix} M_f \end{bmatrix} - \begin{bmatrix} M_n \end{bmatrix})^T (\begin{bmatrix} M_f \end{bmatrix} - \begin{bmatrix} M_n \end{bmatrix})^T
\end{align*}
\] (2.33)

where \( \begin{bmatrix} M \end{bmatrix} \) : lamina local compliance matrix

\( \begin{bmatrix} M_n \end{bmatrix} \) : matrix compliance matrix

\( \begin{bmatrix} M_f \end{bmatrix} \) : fibers compliance matrix

The second step is to get the global thermal strain and stress vectors for each lamina \((i)\). The thermal strain vector in global coordinates can be found by transforming the local thermal strain vector as follows [1.13]:

\[
\mathbf{m}_{global} = [T]^T \mathbf{m}_{local}
\] (2.35)
where ${\mathbf{T}}$ is the transformation matrix expressed in equation (1.27). Hill [1.16] derived the thermal stress vector $\mathbf{t}$ from the stress-strain relations shown below.

\[
\mathbf{e} = \mathbf{[M]}_{\text{global}} \mathbf{\sigma} + \mathbf{m}_{\text{global}} \tag{2.36}
\]

\[
\mathbf{\sigma} = \mathbf{[L]}_{\text{global}} \mathbf{\varepsilon} + \mathbf{t}_{\text{global}} \tag{2.37}
\]

\[
\therefore \mathbf{t}_{\text{global}} = -\mathbf{[T]}_{\text{global}} \mathbf{m}_{\text{global}} \tag{2.38}
\]

On the laminate level, if there are $N$ laminae with thicknesses $t_i$ and the overall laminate thickness is $t$, then using the Classical Lamination theory [1.6], the laminate thermal stress vector is calculated as follows:

\[
\mathbf{t}_{\text{global}} = \frac{1}{t} \sum_{i} t_i \mathbf{t}_{i,\text{global}} \tag{2.39}
\]

If all the laminae have the same thickness, then the above equation becomes:

\[
\mathbf{t}_{\text{global}} = \frac{1}{N} \sum_{i} \mathbf{t}_{i,\text{global}} \tag{2.40}
\]

Finally, the global laminate thermal strain vector is calculated as [1.16]:

\[
\mathbf{m}_{\text{global}} = -\mathbf{[M]}_{\text{global}} \mathbf{t}_{\text{global}} \tag{2.41}
\]

If the global thermal strain vector under plane stress conditions is expressed as:

\[
\mathbf{m}_{\text{global}} = [A \quad B \quad 0]^T, \text{ then } \alpha_t = A, \alpha_r = B \text{ and } \alpha_{rr} = C.
\]
2.4 Tsai-Wu Failure Criterion for Composite Materials

The failure envelope of the Tsai-Wu tensor theory [1.20] is given by:

$$F_1 \sigma_1 + F_3 \sigma_3 + F_4 \sigma_4 + F_{14} \sigma_1^2 + F_{15} \sigma_1^3 + F_{16} \sigma_1^4 + 2F_{13} \sigma_1 \sigma_3 = 1$$  \hspace{1cm} (2.42)

where \( \sigma_1 \) = longitudinal stress in the principal material coordinate system,

\( \sigma_2 \) = out-of-plane axial stress in the principal material coordinate system, that is equal to null under plane stress conditions,

\( \sigma_3 \) = transverse stress in the principal material coordinate system, and

\( \sigma_13 \) = shear stress in the principal material coordinate system.

The factors in equation (2.42) are defined as [1.20]:

$$F_1 = \frac{1}{S_{11r}} + \frac{1}{S_{11c}}$$  \hspace{1cm} (2.43)

$$F_3 = -\frac{1}{S_{11r} S_{11c}}$$  \hspace{1cm} (2.44)

$$F_4 = \frac{1}{S_{33r}} + \frac{1}{S_{33c}}$$  \hspace{1cm} (2.45)

$$F_{14} = -\frac{1}{S_{33r} S_{22c}}$$  \hspace{1cm} (2.46)

$$F_{13} = 0$$  \hspace{1cm} (2.47)

$$F_{13} = -\frac{1}{S_{13s}}$$  \hspace{1cm} (2.48)

The factor \( F_{13} \) cannot be determined from any uniaxial test, instead it can be measured from a biaxial tension test, where \( \sigma = \sigma_1 = \sigma_2 \). It can be evaluated by [1.20]:

$$F_{13} = \frac{1}{2\sigma^2} \left[1 - \left(\frac{1}{S_{11r}} + \frac{1}{S_{11c}} + \frac{1}{S_{33r}} + \frac{1}{S_{33c}}\right) \sigma + \left(\frac{1}{S_{11r} S_{11c}} + \frac{1}{S_{33r} S_{33c}}\right) \sigma^2 \right]$$  \hspace{1cm} (2.49)
For the biaxial tension test shown in Fig. (2.3) above, the strain in the transverse direction is believed to reach the ultimate value first, and it can be expressed as:

\[ \varepsilon_{33T} = \varepsilon_{\text{transverse}} = S_{33T} / E_{33} \]  

(2.50)

also

\[ \varepsilon_{33T} = \varepsilon_{\text{transverse}} = \sigma_{33} / E_{33} - \nu_{13} \sigma_{11} / E_{11} \]

\[ = \sigma \left( t / E_{33} - \nu_{13} / E_{11} \right) \]  

(2.51)

From equations (2.50) and (2.51), \( \sigma \) can be expressed as:

\[ \sigma = S_{33T} / \left[ E_{13} \left( t / E_{33} - \nu_{13} / E_{11} \right) \right] \]  

(2.52)

The procedure of applying this technique to the laminate structure analysis in this study, is to first get the local stresses for each lamina in its principal material directions from the global stresses such that:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = [F]
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix}
\]  

(2.53)

Afterwards, the three local stress components are used in each lamina where the Tsai-Wu criterion is applied. When the first lamina fails, the system is assumed to fail completely. No further investigation is performed on the progressive failure behavior of the laminates.
2.5 Elastic Analysis of Isotropic Concentric Rings with Interferences under Plane Stress Conditions

The following derivation is used for the analysis of the steel ring that is the innermost ring of the concentric rings set, which is used to provide the required interaction with the magnetic bearings. The derivation is also used for the analysis of the concentric aluminum rings case.

The general stress-strain relationships in polar coordinates are included in Appendix (D). For an isotropic material under plane stress conditions, the stress-strain relationships can be written in a matrix form as in equation (2.54) [1.24].

$$
\begin{bmatrix}
\sigma_\theta \\
\sigma_r \\
\tau_{r\theta}
\end{bmatrix} = [L]_{\text{global}} \begin{bmatrix}
\varepsilon_\theta \\
\varepsilon_r \\
\gamma_{r\theta}
\end{bmatrix}
$$

(2.54)

The elastic stiffness matrix can be reduced from 6x6 to 3x3, due to the axi-symmetry of material properties as follows [1.24]:

$$
[L]_{\text{global}} = [L]_{\text{local}} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{bmatrix}
$$

(2.55)

Substituting by equation (2.55) into equation (2.54):

$$
\begin{bmatrix}
\sigma_\theta \\
\sigma_r \\
\tau_{r\theta}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{bmatrix} \begin{bmatrix}
\varepsilon_\theta \\
\varepsilon_r \\
\gamma_{r\theta}
\end{bmatrix}
$$

(2.56)

The general geometrical relationships are included in Appendix (D). For the isotropic case discussed here, the strain-displacement relationships are defined by [1.24]:

$$
\varepsilon_r = \frac{du}{dr}
$$

(2.57)

$$
\varepsilon_\theta = \frac{u}{r}
$$

(2.58)

$$
\gamma_{r\theta} = 0
$$

(2.59)
The following derivation is undertaken to calculate the global stresses due to the centrifugal force of the body created as a result of the constant angular velocity of the rotor. This case has been analyzed in many classical elasticity theory references using the concept of stress function. One of the contributions of this study is the analysis derived below, because it utilizes a new technique for the solution, whose results match exactly those developed using the stress function approach.

Because of the axi-symmetric loading and geometry, the isotropy of the material, and the plane stress conditions, the hoop displacement term \( \nu \) vanishes from the strain-displacement relationships. Substituting by equations (2.57) through (2.59), equation (2.56) becomes:

\[
\begin{align*}
\sigma_r &= \sigma_\theta = \frac{E}{1-\nu^2} \left[ \begin{array}{cc} 1 & \nu \\ \nu & 1 \end{array} \right] \left[ \begin{array}{c} \frac{u}{r} \\ \frac{du}{dr} \end{array} \right] \\
\sigma_r &= \frac{E}{1-\nu^2} \left[ \begin{array}{c} \frac{u}{r} + \nu \frac{du}{dr} \\ \nu \frac{u}{r} + \frac{du}{dr} \end{array} \right] \\
\sigma_\theta &= \frac{E}{1-\nu^2} \left[ \begin{array}{c} \frac{u}{r} + \nu \frac{du}{dr} \\ \nu \frac{u}{r} + \frac{du}{dr} \end{array} \right]
\end{align*}
\] (2.60)

For a rotating disk with constant velocity, the equilibrium equations listed in Appendix (D) reduce to only one equation (2.63).

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r = 0
\] (2.61)

from (2.62),

\[
\frac{d\sigma_r}{dr} = \frac{E}{1-\nu^2} \left[ \frac{\nu du}{r dr} - \nu \frac{u}{r^2} + \frac{d^2 u}{dr^2} \right]
\] (2.64)

Substituting by (2.61), (2.62) and (2.64) in (2.63):

\[
\frac{E}{1-\nu^2} \left[ \frac{\nu du}{r dr} - \nu \frac{u}{r^2} + \frac{d^2 u}{dr^2} \right] = \frac{E}{1-\nu^2} \left[ (\nu - 1) \frac{u}{r^2} + (1 - \nu) \frac{du}{dr} \right] + \rho \omega^2 r = 0
\] (2.65)

Collecting terms:

\[
\frac{E}{1-\nu^2} \left[ \frac{d^2 u}{dr^2} + (\nu + 1 - \nu) \frac{1}{r^2} \frac{du}{dr} + (\nu - 1 - \nu) \frac{u}{r^2} \right] + \rho \omega^2 r = 0
\] (2.66)
\[
\Rightarrow \frac{E}{1 - \nu^2} \left[ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} \right] + \rho \omega^2 r = 0
\]

(2.67)

Simplifying and multiplying both sides by \(r^2\):

\[
\Rightarrow r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - u = -\rho \omega^2 \frac{(1 - \nu^2)}{E} r^3
\]

(2.68)

Equation (2.68) is a second order non-homogeneous total differential equation with variable coefficients, whose solution can not be obtained from straightforward standard solutions. Consequently, Cauchy-Euler method is used as shown in the following steps.

Let \(r = e^t\), \(\frac{du}{dr} = Du\), and \(\frac{du}{dt} = D_t u\), It is proved in Appendix (E) that the following relationships hold:

\(r D_t u = D_r u\)

\(r^2 D_r^2 u = D_r (D_r - 1) u\)

Substituting by the variable \(t\) and its associated differential operators in (2.68),

\[
\Rightarrow D_r (D_r - 1) u + D_t u - u = -\rho \omega^2 \frac{(1 - \nu^2)}{E} e^{\nu t}
\]

(2.69)

Simplifying and collecting terms:

\[
\Rightarrow [D_r^2 - 1] u = -\rho \omega^2 \frac{(1 - \nu^2)}{E} e^{\nu t}
\]

(2.70)

\[
\Rightarrow \left[ \frac{d^2 u}{dt^2} - u \right] = -\rho \omega^2 \frac{(1 - \nu^2)}{E} e^{\nu t}
\]

(2.71)

Equation (2.71) is a second order non-homogeneous differential equation with constant coefficients; its characteristic equation is:

\(m^2 - 1 = 0\)

\((m - 1)(m + 1) = 0\)

\(m_1 = 1\)

\(m_2 = -1\)

From the characteristic equation, the complementary (homogeneous) solution is obtained as:

\[u_c = C_1 e^t + C_2 e^{-t} = C_1 e^t + \frac{C_2}{e^t}\]

(2.72)
where $C_1$ and $C_2$ are the unknown constants of integration.

The particular solution is obtained by assuming a value for it in the form of:

$$u_p = Q_0 e^{3t}$$  \hfill (2.73)

Differentiating once and twice:

$$u_p' = 3Q_0 e^{3t}$$  \hfill (2.74)

$$u_p'' = 9Q_0 e^{3t}$$  \hfill (2.75)

Since the assumed solution must satisfy equation (2.71), then substituting by (2.73), (2.74) and (2.75) into (2.71):

$$\Rightarrow 9Q_0 e^{3t} - Q_0 e^{3t} = -\rho \omega^2 \left( \frac{1-v^2}{8E} \right) e^{3t}$$  \hfill (2.76)

$$\Rightarrow Q_0 = -\rho \omega^2 \left( \frac{1-v^2}{8E} \right) e^{3t}$$  \hfill (2.77)

Now, the overall solution for the radial displacement $u$ is obtained in terms of $t$:

$$u_{\text{overall}} = C_1 e^t + \frac{C_2}{e^t} - \rho \omega^2 \left( \frac{1-v^2}{8E} \right) e^{3t}$$  \hfill (2.78)

The solution as a function of $r$ is:

$$u_{\text{overall}} = C_1 r + \frac{C_2}{r} - \rho \omega^2 \left( \frac{1-v^2}{8E} \right) r^3$$  \hfill (2.79)

To obtain the radial stress, substitute in equation (2.62):

$$\sigma_r = \frac{E}{1-\nu^2} \left[ v \left( \frac{C_1}{r^2} + \frac{C_2}{r} - \nu \frac{1-v^2}{8E} \right) \rho \omega^2 r^2 + C_1 - \frac{C_2}{r^2} - \frac{3}{8} \left( \frac{1-v^2}{E} \right) \rho \omega^2 r^2 \right]$$  \hfill (2.80)

Simplifying and collecting terms:

$$\sigma_r = \left[ \frac{A}{r^2} + C - \left( \frac{v + 3}{8} \right) \rho \omega^2 r^2 \right]$$  \hfill (2.81)

where: $A = -\frac{E}{1+\nu} C_2$ and $C = \frac{E}{1-\nu} C_1$
Similarly, substituting by (2.79) into (2.61):

$$\sigma_\theta = \left[-\frac{A}{r^2} + C - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2\right]$$  \hspace{1cm} (2.82)

Getting \(u\) in terms of A and C:

$$u = \left[\frac{1-\nu}{E} C r - \left(\frac{1+\nu}{E}\right) A - \frac{(1-\nu^2)}{8E} \rho\omega^2 r^3\right]$$  \hspace{1cm} (2.83)

If there are a number of N rings; the innermost ring (1) is made out of steel and the rest of the rings (ring (i)) are made out of aluminum, there are two unknowns per ring. Therefore, there are 2N unknowns, and so, 2N boundary conditions are needed to solve the equations; those are:

- \(r_i\):
  $$\sigma_{r(i)} = 0$$ \hspace{1cm} 1 B.C.

- \(r_{ci}\):
  $$\sigma_{r(i)} = \sigma_{r(i-1)}$$ \hspace{1cm} N-1 B.C.'s
  $$\delta_i = u_{r(i-1)} - u_{r(i)}$$ \hspace{1cm} N-1 B.C.'s

- \(r_o\):
  $$\sigma_{r(N)} = 0$$ \hspace{1cm} 1 B.C.'s

\[\sum\text{B.C.'s} = 2N\]
2.6 Elastic Analysis of Axi-symmetric Laminated Concentric Rings with Interferences under Plane Stress Conditions

There are three cases in which the laminated rings become axi-symmetric with respect to the global coordinate system $\theta$, $z$ and $r$. The importance of axi-symmetry lies in the fact that there is no coupling between extension and twisting effects, i.e., the terms $L_{15}$, $L_{25}$, $L_{51}$ and $L_{52}$ of the global stiffness matrix of a laminate vanish, and hence the analysis can be handled in a more or less straightforward manner. For plane stress conditions, the symmetric global stiffness matrix can be reduced to 3x3 as shown below:

$$
L_{\text{global}} = \begin{bmatrix}
S & T & 0 \\
T & Z & 0 \\
0 & 0 & H
\end{bmatrix}
$$

(2.84)

The three cases where axi-symmetry occur are:

1. The first case is when the lamination angle is zero, therefore the principal material coordinate system coincides with the global coordinate system. In such a case the fibers are aligned in the hoop direction and $S$ is much greater than $Z$.

2. The second case when the lamination angle is 90°, therefore the fibers are aligned in the radial direction; in this case $Z$ is much greater than $S$.

3. The third case when the laminate is made out of a balanced laminae ($\pm \phi$); in this case:

   $$
   S > Z \text{ if } \phi < 45°,
   $$

   $$
   Z > S \text{ if } \phi > 45°,
   $$

   and

   $$
   S = Z \text{ if } \phi = 45° \text{ (Isotropic Case)}
   $$

The following analysis, which attempts to solve for the stresses developed due to the centrifugal effect of the angular velocity of the rotor, can be applied for any of the three cases. This analysis is an essential outcome of this study, because it provides a closed form exact solution for the problem in hand, which has not been treated before in the literature to the best of the knowledge of the author. The only similar derivation found in the literature is of a rotor under plane strain conditions presented by Jeong in [1.5].

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By using the strain-displacement relationships, which can be found in Appendix (D), the stress-strain relationship can be written in a matrix form as:

\[
\begin{bmatrix}
\sigma_\theta \\
\sigma_r \\
\tau_{\theta r}
\end{bmatrix} =
\begin{bmatrix}
S & T & 0 \\
T & Z & 0 \\
0 & 0 & H
\end{bmatrix}
\begin{bmatrix}
\frac{u}{r} \\
\frac{du}{dr} \\
0
\end{bmatrix}
\]

(2.85)

\[\Rightarrow \sigma_\theta = S \frac{u}{r} + T \frac{du}{dr} \]  

(2.86)

\[\Rightarrow \sigma_r = T \frac{u}{r} + Z \frac{du}{dr} \]  

(2.87)

The equilibrium equations in polar coordinates, listed in Appendix (D), reduce to:

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r = 0
\]

(2.88)

From (2.87),

\[
\frac{d\sigma_r}{dr} = \frac{T}{r} \frac{du}{dr} - \frac{T}{r^2} \frac{u}{r} + Z \frac{d^2 u}{dr^2}
\]

(2.89)

Substituting by (2.86), (2.87) and (2.89) in (2.88):

\[\Rightarrow \frac{1}{r} \frac{du}{dr} - \frac{T}{r^2} \frac{u}{r} + Z \frac{d^2 u}{dr^2} + \left( \frac{T - S}{r^2} \right) \frac{u}{r} + \left( \frac{Z - T}{r^2} \right) \frac{1}{r} \frac{du}{dr} + \rho \omega^2 r = 0 \]

(2.90)

Simplifying:

\[Z \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{S}{r^2} \frac{u}{r} = -\rho \omega^2 r \]

(2.91)

Multiplying both sides by \(r^2 Z\):

\[r^2 \frac{d^2 u}{dr^2} + \frac{du}{dr} - \frac{S}{Z} \frac{u}{r} = -\frac{1}{Z} \rho \omega^2 r \]

(2.92)

Equation (2.92) is a second order non-homogenous total differential equation with variable coefficients; therefore, Cauchy-Euler method is used as follows:

Let \(r = e^t\), \(\frac{du}{dt} = Du\), and \(\frac{d^2 u}{dt^2} = D^2 u\)

\[\Rightarrow r Du = D^2 u \text{ and } r^2 D^2 u = D_l(D_l - 1)u \text{ as proved in Appendix (E).} \]

Substitute in (2.92):

\[D_l(D_l - 1)u + D_l u - \frac{S}{Z} u = -\frac{1}{Z} \rho \omega^2 e^t \]

(2.93)
Collecting terms: \((D_i^2 - D_1 + D_1 - \frac{S}{Z})u = -\frac{1}{Z} \rho \omega^2 e^{\frac{\nu}{2}}\) \hspace{1cm} (2.94)

\[\Rightarrow \frac{d^2u}{dt^2} - \frac{S}{Z}u = -\frac{1}{Z} \rho \omega^2 e^{\frac{\nu}{2}}\] \hspace{1cm} (2.95)

Equation (2.95) is a second order non-homogeneous differential equation with constant coefficients, whose characteristics equation is:

\[m^2 - \frac{S}{Z} = 0\]

\[(m - \sqrt{\frac{S}{Z}})(m + \sqrt{\frac{S}{Z}}) = 0\]

\[\Rightarrow m_1 = \sqrt{\frac{S}{Z}} \quad \text{and} \quad m_2 = -\sqrt{\frac{S}{Z}}\]

The complementary (homogeneous) solution for equation (2.95) is:

\[u_c = C_1 e^{\frac{\sqrt{S}}{Z}} + C_2 e^{-\frac{\sqrt{S}}{Z}} = C_1 e^{\frac{\nu}{2}} + C_2 e^{-\frac{\nu}{2}}\] \hspace{1cm} (2.96)

Assume a particular solution of the form:

\[u_p = Q e^{\nu t}\] \hspace{1cm} (2.97)

Differentiating once and twice:

\[u_p' = 3Q e^{\nu t}\] \hspace{1cm} (2.98)

\[u_p'' = 9Q e^{\nu t}\] \hspace{1cm} (2.99)

Substituting by (2.97), (2.98) and (2.99) in (2.95):

\[9Q e^{\nu t} - \frac{S}{Z} Q e^{\nu t} = -\frac{1}{Z} \rho \omega^2 e^{\frac{\nu}{2}}\]

Dividing both sides by \(e^{\nu t}\),

\[\Rightarrow Q(9 - \frac{S}{Z}) = -\frac{1}{Z} \rho \omega^2\]

\[\Rightarrow Q = -\frac{\rho \omega^2}{(9Z - S)}\] \hspace{1cm} (2.100)

From (2.96), (2.97) and (2.100), the overall solution for the radial displacement is expressed as:

\[u_{overall} = C_1 e^{\frac{\nu}{2}} + \frac{C_2}{e^{\frac{\nu}{2}}} - \frac{\rho \omega^2}{(9Z - S)} e^{\nu t}\] \hspace{1cm} (2.101)
The radial displacement is expressed as a function of \( r \) as:

\[
u_{\text{overall}} = C_1 r^\frac{\sqrt{2}}{2} + \frac{C_2}{r^{\frac{\sqrt{2}}{2}}} - \frac{\rho \omega^2 r^3}{(9Z - S)} \quad (2.102)
\]

To get the stresses, differentiate \( u \) once:

\[
\frac{du}{dr} = C_1 \sqrt{\frac{\sqrt{2}}{2}} r^{\frac{3}{2} - 1} - \frac{C_2}{r^{\frac{3}{2} + 1}} - \frac{3 \rho \omega^2}{(9Z - S)} \quad (2.103)
\]

Substituting by (2.102) and (2.103) in (2.87), collecting terms and simplifying:

\[
\sigma_r = \left[ (T + S\sqrt{Z}) r^{\frac{3}{2} - 1} \right] C_1 + \left[ \frac{S - T\sqrt{Z}}{r^{\frac{3}{2} + 1}} \right] C_2 - \frac{(3Z + T)}{(9Z - S)} \rho \omega^2 r^2 \quad (2.104)
\]

Substituting by (2.102) and (2.103) in (2.86), collecting terms and simplifying:

\[
\sigma_\theta = \left[ \frac{S + T}{\sqrt{Z}} r^{\frac{3}{2} - 1} \right] C_1 + \left[ \frac{S - T\sqrt{Z}}{r^{\frac{3}{2} + 1}} \right] C_2 - \frac{(3T + S)}{(9Z - S)} \rho \omega^2 r^2 \quad (2.105)
\]

If there are \( N \) rings: the innermost is made out of steel and the rest are made of a composite material (whether 0° angle, 90° angle or balanced laminates), there will be \( 2N \) unknown coefficients. \( 2N \) boundary conditions are needed to solve for the \( 2N \) unknown coefficients; those are:

\begin{equation*}
\begin{align*}
@ r_i & \quad \sigma_{r(i)} = 0 \quad 1 \text{ B.C.} \\
@ r_{C1} & \quad \sigma_{r(i)} = \sigma_{r(i+1)} \quad \text{N-1 B.C.'s} \\
& \quad \delta_i = u_{r(i+1)} - u_{r(i)} \quad \text{N-1 B.C.'s} \\
@ r_o & \quad \sigma_{r(N)} = 0 \quad 1 \text{ B.C.'s} \\
\end{align*}
\end{equation*}

\[ \Sigma \text{ B.C.'s} = 2N \]
2.7 Elastic Analysis of General Orthotropic Laminated Concentric Rings with Interferences under Plane Stress Conditions

An orthotropic laminate can take several shapes, for instance, a unidirectionally-reinforced angle ply laminate is an orthotropic laminate. The important characteristic of an orthotropic laminate is that its global stiffness matrix is of the form:

\[
[L]_{\text{global}} = \begin{bmatrix}
L_{11} & L_{12} & L_{13} & 0 & L_{15} & 0 \\
L_{21} & L_{22} & L_{23} & 0 & L_{25} & 0 \\
L_{31} & L_{32} & L_{33} & 0 & L_{35} & 0 \\
0 & 0 & 0 & L_{44} & 0 & L_{46} \\
L_{51} & L_{52} & L_{53} & 0 & L_{55} & 0 \\
0 & 0 & 0 & L_{64} & 0 & L_{66}
\end{bmatrix}
\]  \hspace{1cm} (2.106)

It can be noticed that the terms \( L_{15}, L_{25}, L_{31}, \) and \( L_{52} \) are not equal to zero, therefore, there is a coupling between the extension and the twisting effects on an orthotropic laminate. It must also be noticed that the existence of the terms \( L_{44}, L_{46}, L_{64}, \) and \( L_{66} \) creates out-of-plane shearing strain, i.e., \( \gamma_{r\theta} \) and \( \gamma_{\theta\theta} \) are not equal to zero.

The stress–strain relationships under plane stress conditions can be written in a matrix form as:

\[
\begin{bmatrix}
\sigma_{\theta} \\
0 \\
\sigma_{r} \\
0 \\
\tau_{r\theta} \\
0
\end{bmatrix} = [L]_{\text{global}} \begin{bmatrix}
\varepsilon_{\theta} \\
\varepsilon_{r} \\
\varepsilon_{r} \\
\gamma_{r\theta} \\
\gamma_{r\theta} \\
\gamma_{\theta\theta}
\end{bmatrix}
\]  \hspace{1cm} (2.107)

For skew symmetry, which occurs in the general orthotropic laminate form with respect to polar coordinates, \( \nu \neq 0 \) but \( \frac{\partial}{\partial \theta} = 0 \), the strain-displacement relationships are:

\[
\varepsilon_{\theta} = \frac{u}{r}
\]  \hspace{1cm} (2.108)
\begin{align*}
\varepsilon_z &= \frac{\partial \omega}{\partial z} \tag{2.109} \\
\varepsilon_r &= \frac{\partial u}{\partial r} \tag{2.110} \\
\gamma_{n} &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \tag{2.111} \\
\gamma_{r\theta} &= \frac{\partial v}{\partial r} \frac{\nu}{r} \tag{2.112} \\
\gamma_{z\theta} &= \frac{\partial v}{\partial z} \tag{2.113}
\end{align*}

Substituting by equations (2.108) through (2.113) in equation (2.107), we get the following:

\begin{align*}
\begin{bmatrix}
\sigma_{\theta} \\
0 \\
\sigma_{r} \\
0 \\
\tau_{r\theta} \\
0
\end{bmatrix} &= [L]_{\text{global}}
\begin{bmatrix}
\frac{u}{r} \\
\frac{\partial w}{\partial z} \\
\frac{\partial u}{\partial r} \\
\frac{\partial w}{\partial r} + \frac{\partial v}{\partial r} \\
\frac{\partial r}{\partial r} \\
\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}
\end{bmatrix} \tag{2.114}
\end{align*}

Equation (2.114) represents six partial differential equations in two independent variables \( r \) and \( z \), and six unknowns \( u, v, w, \sigma_{\theta}, \sigma_{r}, \tau_{r\theta} \).

The equilibrium equations for this case are:

\begin{align*}
\frac{\partial \sigma_{r}}{\partial r} + \frac{\sigma_{r} - \sigma_{\theta}}{r} + \rho \omega^2 r &= 0 \tag{2.115} \\
\frac{\partial \tau_{r\theta}}{\partial r} + 2 \frac{\tau_{r\theta}}{r} &= 0 \tag{2.116}
\end{align*}
The strategy for solving those first order partial differential equations is to substitute the value of $\partial w / \partial z$ as a function of $\partial u / \partial r$, $u/r$, $\partial v / \partial r$ and $v/r$ from equation (2.114.2) into equations (2.114.1), (2.114.3) and (2.114.5). The next step is to substitute the expressions of $\sigma_\theta$, $\sigma_r$, and $\tau_{\theta \phi}$ as functions of $\partial u / \partial r$, $u/r$, $\partial v / \partial r$ and $v/r$ in the stress equilibrium equations (2.115) and (2.116). Afterwards, the Cauchy-Euler method is going to be used for the separation of variables and for solving for $u$ and $v$ as functions of $r$. $u$ and $v$ are going to be differentiated with respect to $r$ and substituted into equation (2.114.2) to obtain an expression for $\partial w / \partial z$. Considering equations (2.114.2), (2.114.4) and (2.114.6) as a system of partial differential equations, they are going to be solved together by the elimination method to obtain expressions for $u$, $v$ and $w$ as functions of $r$ and $z$. The stresses can then be obtained by substituting into equations (2.114.1), (2.114.3) and (2.114.5).

It has to be noticed that $u$, $v$ and $w$ must be expressed as functions of $r$ and $z$:

Let

- $u = f_1(r)g_1(z)$,
- $v = f_2(r)g_2(z)$, and
- $w = f_3(r)g_3(z)$.

Substitute into equations (2.114.2):

$$\frac{\partial g_3}{\partial z} f_3 = \left[ L_{22} f_1 g_1 + L_{12} g_1 \frac{\partial f_1}{\partial r} + L_{12} \left( \frac{\partial f_2}{\partial r} g_2 - \frac{f_2 g_2}{r} \right) \right] / L_{22}$$  \hspace{1cm} (2.117)

Substituting (2.117) into (2.114.1), (2.114.3) and (2.114.5),

$$\sigma_\theta = \left[ L_{11} - L_{22} \frac{L_{13}}{L_{22}} \right] \frac{f_1 g_1}{r} + \left[ L_{13} - L_{22} \frac{L_{12}}{L_{22}} \right] \frac{\partial f_1}{\partial r} g_1 + \left[ L_{12} - L_{22} \frac{L_{13}}{L_{22}} \right] \frac{\partial f_2}{\partial r} g_2 - \frac{f_2 g_2}{r}$$  \hspace{1cm} (2.118)

$$\sigma_r = \left[ L_{12} - L_{22} \frac{L_{13}}{L_{22}} \right] \frac{f_1 g_1}{r} + \left[ L_{13} - L_{22} \frac{L_{12}}{L_{22}} \right] \frac{\partial f_1}{\partial r} g_1 + \left[ L_{12} - L_{22} \frac{L_{13}}{L_{22}} \right] \frac{L_{12}}{L_{22}} \frac{\partial f_2}{\partial r} g_2 - \frac{f_2 g_2}{r}$$  \hspace{1cm} (2.119)

$$\tau_{\theta r} = \left[ L_{12} - L_{22} \frac{L_{13}}{L_{22}} \right] \frac{f_1 g_1}{r} + \left[ L_{13} - L_{22} \frac{L_{12}}{L_{22}} \right] \frac{\partial f_1}{\partial r} g_1 + \left[ L_{12} - L_{22} \frac{L_{13}}{L_{22}} \right] \frac{L_{12}}{L_{22}} \frac{\partial f_2}{\partial r} g_2 - \frac{f_2 g_2}{r}$$  \hspace{1cm} (2.120)
\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} &= \left[ L_{31} - \frac{L_{32} L_{33}}{L_{22}} \right] g_1 \left( \frac{1}{r} \frac{\partial f_1}{\partial r} - \frac{f_1}{r^2} \right) + \left[ L_{33} - \frac{L_{31} L_{23}}{L_{22}} \right] g_1 \frac{\partial f_1}{\partial r} + \left[ L_{32} - \frac{L_{31} L_{33}}{L_{22}} \right] g_1 \left( \frac{1}{r^2} \frac{\partial f_1}{\partial r} + \frac{f_1}{r^2} \right) \quad (2.121) \\
\frac{\partial \tau_{r\theta}}{\partial r} &= \left[ L_{33} - \frac{L_{32} L_{33}}{L_{22}} \right] g_3 \left( \frac{1}{r} \frac{\partial f_2}{\partial r} - \frac{f_2}{r^2} \right) + \left[ L_{32} - \frac{L_{33} L_{23}}{L_{22}} \right] g_3 \frac{\partial f_2}{\partial r} + \left[ L_{33} - \frac{L_{31} L_{33}}{L_{22}} \right] g_3 \left( \frac{1}{r^2} \frac{\partial f_2}{\partial r} + \frac{f_2}{r^2} \right) \quad (2.122)
\end{align*}
\]

Substituting equations (2.118) through (2.122) into first equilibrium equation (2.115), collecting terms and simplifying:

\[
\begin{align*}
& \left[ L_{23} - \frac{L_{21} L_{23}}{L_{22}} \right] g_1 \frac{\partial^2 f_1}{\partial r^2} + \left[ L_{33} - \frac{L_{31} L_{33}}{L_{22}} \right] g_1 \frac{1}{r} \frac{\partial f_1}{\partial r} + \left[ \frac{L_{33} - L_{11}}{L_{22}} \right] g_3 \frac{f_1}{r^3} \\
& + \left[ L_{23} - \frac{L_{21} L_{23}}{L_{22}} \right] g_2 \frac{\partial^2 f_2}{\partial r^2} + \left[ L_{33} - \frac{L_{31} L_{33}}{L_{22}} \right] g_2 \frac{1}{r} \frac{\partial f_2}{\partial r} + \left[ L_{13} - \frac{L_{23} L_{33}}{L_{22}} \right] g_3 \frac{f_2}{r^3} = -\rho \omega^2 r \quad (2.123)
\end{align*}
\]

Substituting equations (2.118) through (2.122) into second equilibrium equation (2.116), collecting terms and simplifying:

\[
\begin{align*}
& \left[ L_{23} - \frac{L_{21} L_{23}}{L_{22}} \right] g_1 \frac{\partial^2 f_1}{\partial r^2} + \left[ L_{33} + 2L_{23} - \frac{L_{21} L_{33}}{L_{22}} - 2L_{32} L_{23} L_{33} \right] g_1 \frac{1}{r} \frac{\partial f_1}{\partial r} + \left[ L_{33} - \frac{L_{11} L_{33}}{L_{22}} \right] g_3 \frac{f_1}{r^3} \\
& + \left[ L_{23} - \frac{L_{21} L_{23}}{L_{22}} \right] g_2 \frac{\partial^2 f_2}{\partial r^2} + \left[ L_{33} - \frac{L_{31} L_{33}}{L_{22}} \right] g_2 \frac{1}{r} \frac{\partial f_2}{\partial r} + \left[ L_{22} - L_{33} \right] g_3 \frac{f_2}{r^3} = 0 \quad (2.124)
\end{align*}
\]

Let \( A_1 = L_{33} \frac{L_{21}}{L_{22}} \quad B_4 = L_{33} \frac{L_{23}}{L_{22}} \)

\( A_2 = A_1 \)

\( A_3 = L_{21} - L_{11} \quad B_5 = L_{33} \frac{L_{23}}{L_{22}} \)

\( A_4 = L_{33} \frac{L_{21} L_{33}}{L_{22}} \quad B_6 = -B_3 \)

\( A_5 = -A_3 \)

\( A_7 = -\rho \omega^2 \)
Using the coefficients A’s and B’s and multiplying both sides by \( r^2 \), equations (2.123) and (2.124) become:

\[
A_k g_i r^2 \frac{\partial^2 f_1}{\partial r^2} + A_k g_i r \frac{\partial f_1}{\partial r} + A_4 g_i f_1 + A_4 g_i r^2 \frac{\partial^2 f_2}{\partial r^2} + A_4 g_i r \frac{\partial f_2}{\partial r} + A_4 g_i f_2 = A_r r^2
\]  
(2.125)

\[
B_k g_i r^2 \frac{\partial^2 f_1}{\partial r^2} + B_k g_i r \frac{\partial f_1}{\partial r} + B_4 g_i f_1 + B_4 g_i r^2 \frac{\partial^2 f_2}{\partial r^2} + B_4 g_i r \frac{\partial f_2}{\partial r} + B_4 g_i f_2 = 0
\]  
(2.126)

Those are two coupled non-homogenous second order differential equations with variable coefficients. The Cauchy-Euler method is used as follows:

Let \( \frac{\partial}{\partial r} = D \)  
\( \text{i.e.: } \frac{\partial f_1}{\partial r} = D f_1 \)  \( \frac{\partial^2 f_1}{\partial r^2} = D^2 f_1 \)

\( \frac{\partial f_2}{\partial r} = D f_2 \)  \( \frac{\partial^2 f_2}{\partial r^2} = D^2 f_2 \)

Let \( r = e^t \)  \( \frac{\partial}{\partial t} = D_t \)
\( \text{i.e.: } \frac{\partial f_1}{\partial t} = D_t f_1 \)  \( \frac{\partial^2 f_1}{\partial t^2} = D_t^2 f_1 \)

\( \frac{\partial f_2}{\partial t} = D_t f_2 \)  \( \frac{\partial^2 f_2}{\partial t^2} = D_t^2 f_2 \)

The following relations are proved in Appendix D:

\[ r D f_1 = D_t f_1 \]
\[ r D f_2 = D_t f_2 \]

\[ r^2 D^2 f_1 = D_t (D_t - 1) f_1 \]
\[ r^2 D^2 f_2 = D_t (D_t - 1) f_2 \]

Substituting in equations (2.125) and (2.126) and collecting terms:

\[
[A_4 D_t^2 + (A_4 - A_4) D_t + A_4] f_1 g_1 + [A_4 D_t^2 + (A_4 - A_4) D_t + A_4] f_2 g_2 = A_t e^{\gamma t}
\]  
(2.127)

\[
[B_4 D_t^2 + (B_4 - B_4) D_t + B_4] f_1 g_1 + [B_4 D_t^2 + (B_4 - B_4) D_t + B_4] f_2 g_2 = 0
\]  
(2.128)

\[
\Rightarrow f_1 g_1 = \frac{[B_4 D_t^2 + (B_4 - B_4) D_t + B_4]}{[B_4 D_t^2 + (B_4 - B_4) D_t + B_4]} f_2 g_2
\]

Substituting by (2.128) into equation (2.127):

\[
\left[ \frac{[-(A_4 D_t^2 + (A_4 - A_4) D_t + A_4)][B_4 D_t^2 + (B_4 - B_4) D_t + B_4]}{[B_4 D_t^2 + (B_4 - B_4) D_t + B_4]} + [A_4 D_t^2 + (A_4 - A_4) D_t + A_4] \right] f_2 g_2 = A_t e^{\gamma t}
\]  
(2.129)
Using long division:
\[
\left\{-\frac{A_3}{B_3}\left[D_0(D_1^2 + (B_3 - B_4)D_1 + B_6)\right] + \left[A_4D_1^2 + (A_3 - A_4)D_1 + A_6\right]\right\}f_2g_2 = A_4e^{\nu t} \tag{2.130}
\]
Collecting terms:
\[
\left\{A_3 - \frac{A_3B_3}{B_2}\right\}D_1^2 + \left[(A_3 - A_4) - \frac{A_4}{B_3}(B_2 - B_4)\right]D_1 + \left[A_6 - \frac{A_6}{B_3}B_5\right]g_2 = A_4e^{\nu t} \tag{2.131}
\]
Let
\[
S_1 = A_3 - \frac{A_3B_3}{B_2}
\]
\[
S_2 = (A_3 - A_4) - \frac{A_4}{B_3}(B_2 - B_4)
\]
\[
S_3 = A_6 - \frac{A_6}{B_3}B_5
\]
Substituting in (2.131):
\[
\Rightarrow [S_1D_1^2 + S_2D_1 + S_3]f_2 = A_4e^{\nu t}/g_2 \tag{2.132}
\]
\[
\Rightarrow S_1 \frac{\partial^2 f_2}{\partial t^2} + S_2 \frac{\partial f_2}{\partial t} + S_3 f_2 = A_4e^{\nu t}/g_2 \tag{2.133}
\]
Equation (2.133) is a second order non-homogeneous differential equation with constant coefficients. The characteristic equation of equation (2.133) is:
\[
S_1m^2 + S_2m + S_3 = 0
\]
where
\[
m = \frac{-S_2 \pm \sqrt{S_2^2 - 4S_1S_3}}{2S_1}
\]
The complementary (homogeneous) solution is as follows:
\[
f_{2c} = C_1e^{m_1t} + C_2e^{m_2t}
\]
Assuming the particular solution to take the form:
\[
f_{2p} = Q_0e^{\nu t}/g_2
\]
Differentiating once and twice:
\[
f_{2p}' = 3Q_0e^{\nu t}/g_2
\]
\[
f_{2p}'' = 9Q_0e^{\nu t}/g_2
\]
Substituting in equation (2.133):
\[9S_1Q_0 e^{3t} + 3S_3Q_0 e^{3t} + S_3Q_0 e^{3t} \forall g, \frac{A_3}{g};\]
\[\Rightarrow Q_1 = \frac{A_3}{9S_1 + 3S_3 + S_3};\]

The overall solution for \( f_1 \) is:
\[f_1 = C_1 e^{m_1} + C_2 e^{m_2} + Q_0 e^{3t}/g_2\]  
(2.134)

Solving for \( f_1 \) in the same manner:
\[f_1 = C_3 e^{m_1} + C_4 e^{m_2} + Q_0 e^{3t}/g_1\]  
(2.135)

where:
\[m_1 = \frac{-S_3 \pm \sqrt{S_3^2 - 4S_3S_6}}{2S_4}\]

and
\[S_4 = A_4 - \frac{A_4B_4}{B_6};\]
\[S_3 = (A_2 - A_4) - \frac{A_6}{B_6}(B_2 - B_4);\]
\[S_3 = A_5 - \frac{A_6}{B_6}B_2;\]

and \( \Rightarrow Q_1 = \frac{A_3}{9S_4 + 3S_3 + S_3} \)

Transforming from \( t \) to \( r \):
\[f_2 = C_1 r^{m_1} + C_2 r^{m_2} + Q_0 r^3/g_2\]  
(2.136)
\[f_1 = C_3 r^{m_1} + C_4 r^{m_2} + Q_0 r^3/g_1\]  
(2.137)

get \( \frac{\partial f_2}{\partial r} \) & \( \frac{\partial f_1}{\partial r} \)
\[\frac{\partial f_2}{\partial r} = m_1C_1 r^{m_1} + m_2C_2 r^{m_2} + 3Q_0 r^2/g_2\]  
(2.138)
\[\frac{\partial f_1}{\partial r} = m_3C_3 r^{m_1} + m_4C_4 r^{m_2} + 3Q_0 r^2/g_1\]  
(2.139)
Substituting by (2.136) through (2.139) in (2.117) to get a relation between $\partial g_1/\partial z$, $g_1$ and $g_2$, and simplifying:

$$
\frac{\partial g_1}{\partial z} f_2 = -\left\{\left[L_{12} + L_{22}m_3\right]C_3g_1r^{m_{1}+1} + \left[L_{12} + L_{22}m_4\right]C_4g_1r^{m_4-1} + (m_1-1)L_{22}C_1g_1r^{m_1-1} + \left[L_{12} + 3L_{22}\right]Q_xr^2 + 2L_{22}Q_0r^2 \right\}/L_{22} \tag{2.140}
$$

Let

$$
K_1 = f_3
$$

$$
K_2 = \left\{\left[L_{12} + L_{22}m_3\right]C_3r^{m_{1}+1} + \left[L_{12} + L_{22}m_4\right]C_4r^{m_4-1} \right\}/L_{22}
$$

$$
K_3 = \left\{\left(m_1-1\right)L_{22}C_1r^{m_1-1} + \left(m_2-1\right)L_{22}C_2r^{m_1-1} \right\}/L_{22}
$$

$$
K_4 = -\left\{\left[L_{12} + 3L_{22}\right]Q_xr^2 + 2L_{22}Q_0r^2 \right\}/L_{22}
$$

Using the above coefficients, equation (2.140) becomes:

$$
K_1 \frac{\partial g_1}{\partial z} + K_2 g_1 + K_3 g_2 = K_4 \tag{2.141}
$$

Equations (2.114.4) and (2.114.6) can be rewritten as:

$$
L_{44} \left( \frac{\partial g_1}{\partial z} f_1 + \frac{\partial f_1}{\partial r} g_3 \right) + L_{46} \frac{\partial g_1}{\partial z} f_2 = 0 \tag{2.142}
$$

$$
L_{46} \left( \frac{\partial g_1}{\partial z} f_1 + \frac{\partial f_1}{\partial r} g_3 \right) + L_{66} \frac{\partial g_1}{\partial z} f_2 = 0 \tag{2.143}
$$

Let

$$
K_5 = L_{44} f_1
$$

$$
K_6 = L_{44} \frac{\partial f_1}{\partial r}
$$

$$
K_7 = L_{46} f_2
$$

$$
K_8 = L_{46} \frac{\partial f_1}{\partial r}
$$

$$
K_9 = L_{46} \frac{\partial f_1}{\partial r}
$$

$$
K_{10} = L_{66} f_2
$$

Let $\frac{\partial g}{\partial z} = D g$
Equations (1.141), (1.142) and (1.143) become:

\[ K_2 g_1 + K_3 g_2 + K_4 Dg_3 = K_4 \]  
\[ K_5 Dg_1 + K_6 Dg_2 + K_8 g_3 = 0 \]  
\[ K_7 Dg_1 + K_{10} Dg_2 + K_9 g_3 = 0 \]  

Equations (2.144), (2.145) and (2.146) comprises a system of linear differential equations in three dependent variables \((g_1, g_2, \text{ and } g_3)\). The system of equations can be solved together by the elimination method to give explicit expressions for \((g_1, g_2, \text{ and } g_3)\) as functions of \(z\) and functions of \(K_1\) to \(K_{10}\), which are themselves functions of \(f_1\), \(f_2\) and \(f_3\). The solution of the three equations will produce three coefficients of integration. The solutions can then be substituted in equations (2.136), (2.137) and (2.140) to give an exact formulation for \(u\), \(v\) and \(w\), with a total of four unknown constants. \(u\), \(v\) and \(w\) can then be substituted in equations (2.114.1), (2.114.3) and (2.114.5) to obtain a solution for the in-plane hoop, radial and shear stresses. To solve for the four unknown constants of integration, for one ring, the following boundary conditions can be applied:

@ \(r_1\)

\[ \sigma_r = 0 \quad \text{B.C.} \]

\[ \tau_{r\theta} = 0 \quad \text{B.C.} \]

@ \(r_0\)

\[ \sigma_r = 0 \quad \text{B.C.} \]

\[ \tau_{r\theta} = 0 \quad \text{B.C.} \]

\[ \Sigma \text{B.C.'s} = 4 \]

It has to be mentioned that the exact solution for such problem is very complicated as can be seen from the steps laid out above. Therefore, using the Finite Element Technique would be more efficient and less expensive in such a case, although the geometry is a very simple one.
2.8 Modeling

The solutions obtained for the isotropic rotor as well as the axi-symmetric laminated rotor cases presented in sections (2.5) and (2.6) are closed form solutions; whereas, the solution obtained for the axi-nonsymmetric case is a numerical one. MATLAB programs were prepared to perform the calculations of the isotropic and axi-symmetric cases, whereas the Finite Element Package I-DEAS was used to perform the calculations of the axi-nonsymmetric case. A MATLAB Program was also prepared to perform the calculations of the increase in temperature needed for the shrink-fitting of concentric rings rotor.

2.8.1 Program Structure for Isotropic and Axi-symmetric Cases

MATLAB professional version 4.2B was used to prepare programs to calculate the stresses and displacements generated for each case at different angular velocities. A sample program for one-steel-five-GRE rings rotor can be found in Appendix (F). The structure of the program is shown in Fig. (2.4). The program structure is a very simple one. It follows exactly the steps explained in the analysis. The program consists basically of four parts. The first part calculates the overall properties of the five balanced GRE laminated rings in global coordinates. The input to this part of the program is the amounts of reinforcement volume fraction and the lamination angle of each ring.

The second part of the program calculates the depth and the angular velocity of the rotor. In this part of the program, the inner, outer and contact radii and the target mass of the rotor are inputted. Using those radii and the calculated densities of each ring, a specific rotor depth is calculated based on the inputted target mass. The mass moment of inertia of the rotor is also calculated and the required angular momentum is inputted to calculate the required angular velocity.

The third part of the program calculates the displacements and the stresses in all rings. It starts by using the output of the first part in calculating the roots of the characteristic equations of the homogenous part of the rings’ displacements. The program then calculates the particular part of the solutions of rings’ displacements. The input to this part of the program is the amounts of interferences at the five contact radii. Using this
input with the calculated velocity from the second part of the program, and using the boundary conditions, the coefficients of integration for each ring are calculated. The radial and hoop stresses and radial displacements are then calculated for each ring. The values of the radial stresses are checked at the contact radii, if they are positive, the amounts of the interferences are re-inputted and should be increased, and the calculations of the third part are repeated.

The last part of the program calculates the in-plane strengths of the five laminated rings using the reinforcement volume fractions inputted in part one. Tsai-Wu failure criterion factors are then calculated. The maximum hoop stress and the radial stress at the same point are calculated from the output of part three. The factor of safety is then found by using the calculated factors and the maximum stresses.
Fig. (2.4) Isotropic/Axi-symmetric Cases Program Structure
2.8.2 Finite Element Modeling for Axi-nonsymmetric Case

The axi-nonsymmetric case is modeled using the Finite Element package I-DEAS version 6. The geometry is very simple, consists of a disk of 45 mm inner radius, 150 mm outer radius, and 16.67 mm length. The rotor mass is 2 kg, and its density is 1830 kg/m³. The part was created by first generating a solid disk from a circle of 150 mm radius that is extruded to a 16.67 mm length. Partitions were used to divide the solid disk into four quadrants; this is needed to create a fine parametric mesh. A cut out was made through the solid disk to create a hole with a 45 mm radius.

The element type that was used in the mesh is an 8-noded solid brick element. The material properties inputted to the model are of a 60% graphite-reinforced-epoxy with 20° lamination angle with respect to the global hoop direction. This is a fully orthotropic material with 9 independent properties. The properties with respect to the global polar coordinate system are calculated using the Mori-Tanaka model [1.13] and the Classical Lamination Theory [1.6] described in chapter one. The properties are:

\[
E_o = 25.126 \text{ GPa} \quad E_r = 4.8386 \text{ GPa} \quad E_z = 4.5394 \text{ GPa}
\]

\[
\nu_{or} = 0.0622 \quad \nu_{rr} = 0.4109 \quad \nu_{rz} = 0.3042
\]

\[
G_{or} = 3.674 \text{ GPa} \quad G_{rz} = 1.676 \text{ GPa} \quad G_{rr} = 2.674 \text{ GPa}
\]

The meshing is shown in Fig. (2.5). The parametric meshing consists of 1200 elements distributed as follows: 40 elements along the inner and outer radii, 10 elements along the radial line from inner to outer radii, and 3 elements across disk length. The elements’ local coordinate systems are referenced to the global polar coordinate system, in which the material properties were defined.

In reality, the rotor rotates freely on the magnetic bearings, and it is held in place by magnetic forces. To model this behavior, rigid body motion should be prevented and constant angular velocity should be specified as a body force. This is done by constraining 3 nodes on the disk to constrain the whole body in 6 degrees of freedom. As seen in Fig. (2.5), two nodes on the inner surface along the z-axis are constrained, one in r, \( \theta \) and z directions and the other is constrained in r and \( \theta \) only. A third node on the
circumference along the radial line of the node constrained in $r$ and $\theta$ is constrained in $\theta$. The loading applied is a body force in the form of angular velocity around the $z$-axis with a magnitude of 25,641 rpm, which gives an output angular momentum of 55 Nms.

Finally, linear elastic analysis was used to generate a solution for the displacements, strains and stresses. The post-processing module was used to display the generated results.

Fig. (2.5) Axi-nonsymmetric FEM Mesh and Boundary Conditions
2.8.3 Calculation of Temperature Increase Needed for Shrink Fits

The value of the interference applied at a specific contact radius is limited by the increase in temperature needed to be applied to create this interference. The limitation stems from the fact that Epoxy is a thermoset plastic, which starts to get damaged at a temperature of 180°C [1.25]. Therefore if the rings are going to be heated from a temperature of 20°C, the maximum interference which can be reached using the shrink-fitting method, is the one that is created at a temperature of 170°C. Therefore, a maximum of 150°C increase in temperature is allowed. A MATLAB program was created to check that the amount of interferences required by the design do not require an increase in temperature more than 150°C. A sample program for one-steel-five-GRE axi-symmetric rings rotor can be found in Appendix (G). The structure of the program is shown below.

![Diagram](Fig. (2.6) Temperature Increase Calculations Program Structure)
2.9 Investigated Parameters

Before listing the parameters that will be investigated in each case of the proposed rotor design, be it the isotropic aluminum case, the axi-symmetric GRE case or the axi-nonsymmetric GRE case, it has to be mentioned that the applied speed depends on the rotor overall target mass. This is because the inner and outer radii of the rotor are fixed for all cases (45 and 150 mm respectively). Therefore when a specific mass is assigned, the densities are calculated based on reinforcement volume fraction in the GRE case, and the rotor length is determined based on the target mass. Consequently for a rotor made from aluminum or GRE, with the same mass, their lengths will be different based on the densities, whereas their angular velocities would be the same to give the same required maximum angular momentum, which is 55 Nms.

First of all the performance of a conventional magnetic bearings wheel is going to be studied. The newly introduced magnetic bearings momentum wheels are made from one inner steel ring and one outer aluminum ring, with spokes joining them. However, the design discussed here is a similar design of two concentric full rings, used for simplicity; the narrow inner ring is made out of steel and the wide outer ring is made out of aluminum alloy 2014-T6. At a target mass of 2 kg, the stresses and the displacements are calculated and the design factor of safety is calculated based on von Mises Failure Envelope.

Secondly, a comparison between the performance of one-steel-five-aluminum rings rotor and one-steel-five-GRE axi-symmetric rings rotor is presented. For the same mass, the performance, in terms of amounts interferences needed, is studied. The resulting hoop stresses at different speeds are also compared for the two cases. Finally, the maximum speed that can be applied for each rotor before failure is calculated.

Parametric analysis of the elements of the design of the axi-symmetric GRE rotor case is carried out. Four parameters are investigated in this case. Firstly, two strategies for the selection of the amount of interferences needed to keep the rings in contact while rotating at maximum speeds are studied; one strategy suggests the use of uniform amounts of interferences at all contact radii, while the other suggests the use of increasing
amounts of interferences with larger radii. Secondly, the effect of the amount of
reinforcement volume fraction is studied, in terms of the resulting hoop stresses and radial
displacements. Thirdly, the effect of changing the lamination angle in a balanced laminate
system is investigated. Finally, the effect of varying the number of rings on the rotor
design, using same inner and outer radii is analyzed.

The case of axi-nonsymmetric rotor is then investigated. A discussion on its
closed form solution is presented and then the Finite Element Model results are shown.
Finally, suggested optimized parameters for the design of the FMW rotor are presented
with lowest possible mass.
Chapter Three

Results & Discussion

This part of the study is divided into five main sections. The first one presents the results of the analysis of a conventional rotor made from steel and aluminum rings. The second part presents a comparison between the performance of a conventional rotor made from steel and aluminum rings and another one made from steel and graphite-reinforced-epoxy laminated rings. The third part provides the parametric optimization analysis of the axi-symmetric laminated concentric rings variables including: reinforcement volume fraction, lamination angle, amount of interferences, and number of laminated rings. The fourth part of the chapter includes the results of the finite element analysis of the axi-nonsymmetric laminated concentric rings rotor design. Finally, the last part presents the suggested design parameters for the FMW rotor under study.

3.1 Conventional Isotropic Steel-Aluminum FMW Rotor Performance

The results presented in this section are based on the analysis undertaken in section (2.5). This analysis uses the Cauchy-Euler method to solve the variable-coefficients second order differential non-homogenous equilibrium equation. The closed form solution of this method is exactly the same as the conventional stress function approach solution that has been used extensively in the theory of elasticity references.

For the desired angular momentum storage of 55 Nm, and for a reduction of rotor mass from 4 kg to 2 kg, the speed needed is 25,641 rpm. The steel ring inner diameter is 45 mm, the contact radius is 50 mm, and the aluminum ring outer radius is 150 mm. The inner and outer radii specified are the same as those of the actual FMW rotor. The rotor length is 10.99 mm for a mass of 2 kg. At this speed, 0.25 mm of interference is needed between the aluminum and the steel rings to keep them at a contact pressure of -2.9 MPa as shown in Fig. (3.1). Figures (3.2) and Fig. (3.3) show the hoop stress distribution and the radial displacement of the rotor respectively. Notice that the maximum hoop stress occurs in the aluminum ring at the contact surface, and it is equal to 380 MPa, which is
very close to the yield strength of the aluminum alloy, that is 410 MPa. According to von Mises failure criterion, the factor of safety for the rotor design at this speed is only 1.15, which is quite low.

![Radial Stress (SigmaR) in All Rings](image)

**Fig. (3.1) Radial Stress for a 1-Steel-Ring 1-Aluminum-Ring Rotor @ 25,641 rpm**
Fig. (3.2) Hoop Stress for a 1-Steel-Ring 1-Aluminum-Ring Rotor @ 25,641 rpm

Fig. (3.3) Radial Displacement for a 1-Steel-Ring 1-Aluminum-Ring Rotor @ 25,641 rpm
3.2 Comparison between the Performance of Aluminum Rings and Graphite-Reinforced-Epoxy Rings in the Design of FMW Rotor

As was mentioned before, the innermost ring should be made out of steel so as to provide the required interaction with the magnetic bearings. The rest of the rings could be made out of aluminum or graphite-reinforced-epoxy (GRE) laminates, depending on their performance, whichever is better. This part of the study provides an answer for this question. The first and simplest comparison to be made, is to have same design parameters for both cases (aluminum and GRE), and compare the stresses and strains or displacements in each case.

If the rotor is made out of one steel ring and 5 aluminum or GRE rings, and the mass is fixed to be 2 kg, the angular velocity in such a case is 25,641 rpm, which gives the required output momentum of 55 Nms. The geometry is also the same, the inner, contact, and outer radii are 45, 50, 70, 90, 110, 130 and 150 mm respectively. For a 2 kg mass, the length of the aluminum rotor is 10.99 mm. Two reinforcement volume fractions in the GRE case are used: 30% and 60%. For a volume fraction of 60% graphite, the GRE rotor is 15.527 mm long, and for a volume fraction of 30% graphite, the GRE rotor is 15.734 mm long. For the sake of comparing of the performance of the two materials, the lamination angle used is zero, which means that the fibers are aligned in the hoop direction. The interferences between the rings are designed such that the contact normal pressure at the interface between the 5 rings under study (aluminum or GRE) is minimal. Just to guaranteed contact between the rings at the operating speed considered. This is interpreted here as:

\[ \sigma_r = 0.5 \text{ MPa} \]

The values of the interferences needed for each case, the resulting hoop stresses and radial displacement are then compared for the performance evaluation. The radial stress distribution of the three materials are shown in Fig. (3.4).
Fig. (3.4) Radial Stress in a 6-Ring Rotor with Minimal Interferences @25641 rpm

The interferences required to attain such radial stress distributions are shown in Fig. (3.5). It can be noticed that in terms of performance based on radial stress distribution, the interferences required for the aluminum rings are on the average five times those required for the 30% graphite-reinforced-epoxy rings, and ten times those required for the 60% graphite-reinforced-epoxy rings.
Fig. (3.5) Comparative Amounts of Interferences for a 6-Ring Rotor @25,641 rpm

The hoop stress distribution associated with the interferences plot above are shown in Fig. (3.6) below. It can be shown that the hoop stresses in the aluminum case reach 425 MPa in the outermost ring whereas it is only 275 MPa at the same point for the 30% and 60% graphite-reinforced-epoxy cases. It should be noted that at this speed, the aluminum rotor will start yielding at the outermost contact surface. That is because the aluminum yield strength is 410 MPa (Aluminum Alloy 2014-T6) [27]. The radial displacements for the three cases shown in Fig. (3.7) indicate that the maximum radial displacement of the aluminum rotor exceeds 4 times that of the 30% GRE rotor, and 8 times that of the 60% GRE rotor.
Fig. (3.6) Hoop Stress in a 6-Ring Rotor with Minimal Interferences @25,641 rpm

Fig. (3.7) Radial Displacement in a 6-Ring Rotor with Minimal Interferences @25,641 rpm
Another comparative point of interest which shows the advantage of using GRE over conventional aluminum rotors is the resulting maximum hoop stresses in both rotors at same speeds. For this comparison, 50% reinforcement volume fraction is used with a lamination angle of zero for all the GRE rings. For both the aluminum as well as the GRE rotors, a mass of 2 kg is used. Figure (3.8) shows the comparative maximum hoop stresses in each rotor at speeds of 5,000, 10,000, 15,000 and 20,000 rpm, with implementation of minimal interferences strategy, that is ensured by selecting the amount of interferences which makes the radial stresses do not decrease than -0.5 MPa at the interfaces. It is worth mentioning that at all the speeds investigated in Fig. (3.8), the maximum hoop stresses developed in the aluminum rings rotor are approximately 1.5 times those developed in the 50% GRE rings rotor, which is a result of the high stiffness of the graphite fibers.

![Maximum Hoop Stresses](image)

**Fig. (3.8) Comparative Amounts of Maximum Hoop Stresses for a 6-Ring Rotor**
The amounts of interferences which are used to attain the behavior shown in Fig. (3.8) are much higher for the aluminum rings rotor than those needed for the 50% GRE rings rotor. The average factors by which the amount of interferences of the GRE rotor are multiplied by to be equal to those of the aluminum rotor interferences, are calculated and shown in Fig. (3.9). To get the average interference multiplication factor at any speed, the ratios of the amount of interferences needed for the aluminum rotor at the interface radii from 2 to 5, to those needed for the GRE rotor are calculated and then averaged. The values of the amount of interferences at the most inner interface is not used since the amount of interferences needed at this inner radius is very small, especially for the very stiff GRE rotor. From Fig. (3.9), it can be noticed that the amount of interference needed for the aluminum rotor at the speeds of 10,000, 15,000, and 20,000 rpm are on the average 7.6 times those needed for the GRE rotor. However, the average interference multiplication factor at the speed of 5,000 rpm is only 5.4. This is due to the fact that this is a comparatively low speed and the amount of interferences used for both rotors are very small.

![Average Interference Multiplication Factor for Different Speeds](image_url)
One final point of comparison between an aluminum rotor and a 50% GRE rotor is the maximum speed each one can reach at a safety factor close to unity, which means the speed each can reach before failure. The failure criteria used for the isotropic aluminum rotor is Von Mises criterion, whereas, for the GRE rotor, Tsai-Wu criterion is applied. It was found out that for a safety factor of 1.04, the maximum speed the aluminum rings rotor can reach is 25,000 rpm, whereas the 50% GRE rings rotor can reach up to 45,000 rpm. Form Fig. (3.10), it is noticed that at those ceiling speeds and limiting factor of safety, the GRE rotor can withstand much higher hoop stresses than the aluminum rotor; which is mainly due to the high tensile strength of the graphite fibers. To reach the 25,000 rpm, the aluminum rotor needs around 2.6 times the amount of interferences used for the GRE rotor at 45,000 rpm.

![Graph](image_url)

Fig. (3.10) Hoop Stress Distribution at Maximum Speeds before Failure
3.3 Parametric analysis of axi-symmetric concentric rings

Rotor design variables

This section is based on the analysis presented in section 2.6. As mentioned before, while designing a rotor that is supported by magnetic bearings, it should be taken into consideration that the inner ring should be made out of steel to provide the required interaction with the magnetic bearings. The rest of the concentric rings are made here from axi-symmetric laminates. The design of the rotor in such a case has four parameters which are analyzed and optimized in this section. The solution of this type of problems is derived and presented in a closed form in section (2.6). In this section, four variables of the axi-symmetric laminated rings rotor are investigated; those are: amount of interferences, number of laminated rings, reinforcement volume fraction, and lamination angle.

3.3.1 Effect of the amount of interferences on rotor performance

The amount of interferences between neighboring rings is an important element in determining the stress distribution in the rotor. Two main criteria of allocating the amounts of interferences are investigated here: the first is using equal amounts of interferences at all interfaces, and the second is using larger amounts of interferences at larger contact radii. For both criteria, two precautions have to be taken. First, it is important to make sure that the amount of interference chosen at any interface would keep the rings in contact at the maximum operating speed of the rotor. Second, since the interferences between the rings are going to be created by shrink-fitting the rings, it is important to check that the amount of interferences chosen can be created by heating the rings to a temperature well below the damage temperature of the resin, which is 180° [25].

The first criterion in allocating the amounts of interferences between the rings is to use equal interferences. Figure (3.1) shows the radial stress distribution of a 1-steel-ring and a 5 GRE rings, with 0° lamination angle and 60% reinforcement 2 kg rotor using equal interferences at the 5 interfaces. The amounts of interferences used are 0.1 mm, 0.06 mm, and 0.03 mm. Notice that for the 0.03 mm interference, and at this speed of 25,641 rpm, σ,
at the interface between ring 5 and 6 is approximately equal to zero, which means that they are about to leave each other, therefore, this is the minimum interference that can be used.

![Radial Stress (SigmaR) in All Rings](image)

**Fig. (3.11) Radial Stress of a 6-Ring Rotor using Different Values of Equal Interferences**

The hoop stress distributions associated with the above chosen interferences are shown in Fig. (3.12) below. It is obvious that for smaller interferences, the rotor hoop stress profile is much better. By decreasing the interferences from 0.1 mm to 0.03 mm, the maximum tensile stress of the outer ring decreased from 310 MPa to 240 MPa, and the compressive hoop stresses in the inner steel ring went from -250 MPa to only -35 MPa. The radial displacements for the three cases are shown in Fig. (3.13), and the same behavior is observed. Therefore, it is advised to use minimum interferences that would still keep the rings in contact at maximum operating speeds.
Fig. (3.12) Hoop Stress of a 6-Ring Rotor using Different Values of Equal Interferences

Fig. (3.13) Radial Displacement of a 6-Ring Rotor using Equal Interferences
The second criterion of allocating the amounts of interferences is to increase them gradually as the contact radii increases. The main problem about this criterion is that this increase should be done based on the idea that minimum contact pressures are kept, hence the minimum amounts of interferences are found by trial and error. Such a radial stress distribution is shown below in Fig. (3.14) along with the interferences used for a 1 steel-ring and a 60% 0° angle GRE S ring 2 kg rotor operating at 25,641 rpm to give angular momentum of 55 Nms. Figure (3.15) compares the hoop stresses resulting from the interferences indicated in Fig. (3.14) to the hoop stresses resulting from using an interference of 0.03 mm at all interfaces, which is the minimum amount of equal interferences that can keep all the rings in contact at this operating speed.

![Radial Stress (SigmaR) in All Rings](image)

Fig. (3.14) Radial Stress of a 6-Ring Rotor using Different Values of Interferences
Fig. (3.15) Hoop Stress Distributions of a 6-Ring Rotor @25,641 rpm

There are two advantages for adopting minimum radial stresses strategy to determine interferences at different interfaces over using equal interferences strategy. The first advantage is that with small interferences implemented at small radii, lower temperatures are needed for shrink-fitting. The second advantage is that although the maximum tensile hoop stress in the outermost ring is the same in both cases, the hoop stress distribution in all the other rings is more homogenous with the variable interferences strategy.
3.3.2 Effect of Reinforcement Volume Fraction on Rotor Performance

While the increase of reinforcement content in a lamina is costly, it has two favorable effects on the behavior of the material. An increase in the reinforcement volume fraction results in an increase in the overall material modulii and strength, which does not mean that the material is only more stiff against forces acting on it, but also can withstand higher stresses. To demonstrate those advantages of increasing the reinforcement volume fraction, the stress distribution resulting from two volume fractions 30% and 60° 0° angle laminates are investigated. For those two volume fractions, equal interferences can not be used because this will induce higher -not needed- stresses in the laminates with higher reinforcement volume fractions. Using the two strategies of allocation of amounts of interferences mentioned in section (3.3.1), the same radial and hoop stress distributions are created and the amounts of interferences required in each case are compared.

Using the strategy of equal interferences, the radial and hoop stress distributions of 60% GRE with interference of 0.03 mm at all interfaces, and with interferences of 0.06 mm at all interfaces, and 30% GRE with interferences of 0.06 mm at all interfaces are shown in Fig. (3.16) and Fig. (3.17) respectively. Notice that for half reinforcement volume fraction, double the amount of interferences is needed. It can be noticed also that for the case of 60% GRE with interferences of 0.06 mm, the material is overstressed.
Fig. (3.16) Radial Stress of a 6-Ring Rotor with 60% and 30% GRE (Equal Interferences)

Fig. (3.17) Hoop Stress of a 6-Ring Rotor with 60% and 30% GRE (Equal Interferences)
Using the second (variable) interferences strategy (minimum contact pressure) for the 30% and 60% GRE, the same radial and hoop stress distributions are obtained as shown in Fig. (3.18) and Fig. (3.19) respectively. Notice again that the amount of the last three interferences of the 30% GRE rotor are double those needed for the 60% GRE rotor.

Although in both cases discussed above (equal and variable interferences), the stress distributions are almost identical, and the interferences required for the 30% GRE are almost double those needed for the 60% GRE, a very important point of comparison is still not shown, that is the factor of safety associated with using both volume fractions in the design of the rotor. Using Tsai-Wu failure criterion, it is found out that in the case of 30% GRE rotor, the F.S. is 1.48 for both types of interferences strategies; whereas, for the 60% GRE, the F.S. is 2.61 for both types of interferences strategies. All the above comparisons show the effect of reinforcement volume fraction on the stiffness of the material, while the last factor-of-safety-based comparison shows the effect of reinforcement volume fraction on overall composite material strength.
Fig. (3.18) Radial Stress of a 6-Ring Rotor with 60% and 30% GRE (Variable Interferences)

Fig. (3.19) Hoop Stress of a 6-Ring Rotor with 60% and 30% GRE (Variable Interferences)
3.3.3 Effect of the Lamination Angle on Rotor Performance

The lamination angle could vary from 0°, which is the case when the fibers are perfectly aligned in the hoop direction, to 90°, which is the case where the fibers are aligned in the radial direction. Figure (3.20) shows the radial stress distribution of a 60% GRE at 0° ply angle and at ±20° ply angle (balanced laminate). The interferences needed for the ±20° laminated rotor to have the shown radial stress distribution is 6 times those needed for the 0° laminated rotor. Figure (3.21) shows the hoop stress distribution for both lamination angles. The maximum hoop stress for the ±20° ply angle laminates exceeds that of the 0° ply angle laminates by 75 MPa. Furthermore, the hoop stress distribution for the 0° ply angle laminates is more uniform in the inner rings. Consequently, it can be concluded that for such an application, the fibers should be aligned in the hoop direction to increase the stiffness in that direction so as to minimize the high stresses resulting from the centrifugal effect.

![Radial Stress (SigmaR) in All Rings](image)

**Fig. (3.20) Radial Stress Distribution for a 6-Ring Rotor @ 25,641 rpm**
Hoop Stress (SigmaTheta) in all Rings

1: 60% GRE @ 0 degree angle
2: 60% GRE @ +/-20 degrees angle

Fig. (3.21) Hoop Stress Distribution for a 6-Ring Rotor @ 25,641 rpm

The amount of interferences needed in each case are: for the 0° laminated rotor, 0.0001, 0.008, 0.015, 0.024, and 0.035 mm, and for the case of ±20° laminated rotor the interferences are 0.032, 0.0494, 0.0903, 0.148 and 0.205 mm.
3.3.4 Effect of the Number of Laminated Rings on Rotor Performance

For the same inner and outer radii, the rotor can be divided into only one inner steel ring and one axi-symmetric laminated ring, or one steel ring and a number of axi-symmetric laminated rings. The rationale behind dividing the laminated ring into a number of rings is to create interferences between the rings, which help in reducing the high hoop stresses induced from the high angular velocity of the rotor. This technique allows also for variation of reinforcement volume fraction and lamination angle from ring to another. This section presents the effect of the number of rings on the stress and displacement levels of the rotor. The inner and outer radii of the steel ring are 45 and 50 mm respectively, and the rotor outer radius is 150 mm. The laminated part of the rotor is studied when it is composed of 2 and 5 rings. Fig. (3.22) shows the radial stress distribution for a 3-ring rotor and a 6-ring rotor. The interferences needed for the 3-ring rotor are 0.0001 and 0.026 mm at inner to outer contact radii respectively. The interferences needed for the 6-ring rotor are 0.0001, 0.0075, 0.0148, 0.0235 and 0.0345 at contact radii starting from the innermost one.

Fig. (3.22) Radial Stress Distribution for a 3-Ring and a 6-Ring Rotors @ 25,641 rpm
Figures (3.23) and (3.24) show the hoop stress distribution and the radial displacement for both rotors. The maximum hoop stress for the 6-ring rotor is 22% higher than that of the 3-ring rotor, whereas the maximum radial displacement of the 6-ring rotor is 12.5% higher than that of the 3-ring rotor. At this speed, the factor of safety for the 6-ring rotor is 2.6, while it is 2.7 for the 3-ring rotor. Although the idea of the use of shrink-fitted rings is of good impact on decreasing the hoop stresses in the inner rings significantly, the outer ring always suffers from exaggerated stresses; therefore, it can be concluded that the fewer the number of rings used, the less the maximum hoop stress becomes. This conclusion opposes Jeong’s proposal [1.5] for a similar rotor design, which suggests the use of ten concentric rings for a rotor with inner and outer radii of 50 and 100 mm respectively. Jeong indicates in his proposal that using larger number of rings decreases the overall stresses in the rotor. The idea of using larger number of shrink-fitted rings to attain higher speeds was also suggested by Kirk in an attempt to optimize the performance of a rotor used for an energy storage system [1.4].

![Hoop Stress (SigmaTheta) in all Rings](image)

Fig. (3.23) Hoop Stress Distribution for a 3-Ring and a 6-Ring Rotors @ 25,641 rpm
The hoop stress distribution of the 3-ring rotor shown in Fig. (3.23) indicates that the behavior of an axi-symmetric laminate with the angular speed as the only loading, is different from that of an isotropic material. This is clearly shown in Fig. (3.23), as the maximum hoop stress occurs at a point close to the outer radius of the outer ring. For an isotropic material, the hoop stress is always maximum at inner radius. The behavior of the axi-symmetric laminate is clear in the 3-ring rotor while it is not in the 6-ring rotor, which might be misleading, however it can be justified. It can be shown that for the 3-ring rotor, the hoop stress is high at the inner radii of both the axi-symmetric rings (rings 2 and 3), because of the interference applied, the hoop stress then decreases for part of the radii and then goes to a maximum at a point close to the outer radius. For the 6-ring rotor, because the rings have shorter radii, the interferences occur at points where the hoop stresses are decreasing before they rise, therefore this phenomenon is not shown in the 6-ring rotor.
Arriving at the above conclusion, it is important to well-understand the behavior of an axi-symmetric laminate under centrifugal force loading without interferences so as to be able to reach an optimized design for the rotor. Figures (3.25) and (3.26) show the radial and hoop stress distributions of a one axi-symmetric ring rotor with five different lamination angles: 0°, (±10°), (±20°), (±30°), and (±40°). The rotor mass is 2 kg and it rotates at 25,641 rpm to give angular momentum of 55 Nms.

![Radial Stress (SigmaR)](image)

Fig. (3.25) Radial Stress Distribution for a 1-Ring Rotor with different Lamination Angles
Fig. (3.26) Hoop Stress Distribution for a 1-Ring Rotor with different Lamination Angles

It can be seen that the lowest maximum hoop stress occurs in the (±20°), axi-symmetric laminate, and it is equal to 185 MPa. It is important to notice that for the 0° and (±10°), axi-symmetric laminates, the hoop stresses are maximum at a point close to the end of outer radius of the ring. For the (±30°), and (±40°), axi-symmetric laminates, the hoop stresses are maximum at the inner radius of the ring. The hoop stress distribution for the (±40°), axi-symmetric laminated ring is very close to that of an isotropic material; that is because at the angle, the material is nearly isotropic in behavior because the Young’s modulus in the hoop direction is close to that in the radial direction. For a (±40°), axi-symmetric laminate, the two modulii are equal.
3.4 Finite Element Model Results for Axi-nonsymmetric Rotor Design

The radial, hoop and in-plane shear stress distributions for the Axi-nonsymmetric one ring rotor are shown in Fig. (3.27), Fig. (3.28) and Fig. (3.29) respectively. Figures (3.30), (3.31) and (3.32) show the radial, hoop and axial displacements. The rotor is 60% GRE with +20° lamination angle. It is a 2 kg rotor, rotating at 25,641 rpm. The results shown below are expressed in SI units.

Fig. (3.27) Radial Stress Distribution for an Axi-nonsymmetric 1-Ring Rotor @25,641 rpm
Fig. (3.28) Hoop Stress Distribution for an Axi-nonsymmetric 1-Ring Rotor @25,641 rpm

Fig. (3.29) In-Plane Shear Stress Distribution for an Axi-nonsymmetric 1-Ring Rotor @25,641 rpm
Fig. (3.30) Radial Displacement and Deformed Shape for an Axi-nonsymmetric 1-Ring Rotor

Fig. (3.31) Hoop Displacement and Deformed Shape for an Axi-nonsymmetric 1-Ring Rotor
Fig. (3.32) Axial Displacement and Deformed Shape for an Axi-nonsymmetric 1-Ring Rotor

Figure (3.27) shows that the radial stress is maximum at a point approximately at mid span of the radial line, and it is equal to 38.8 MPa, which is expected. Although the minimum radial stresses are located at the inner and outer radii, they are not exactly equal to zero, which can be due to numerical errors in the solution. This might also be attributed to the fact that the appearing values of the stresses at the outer surface are the averages of the stresses of the nodes surrounding the surface.

Figure (3.28) shows that the maximum hoop stress occurs at the inner radius and it is equal to 158 MPa. The hoop stress decreases outwards along the radius; however it can be seen that it rises a little bit from 140 MPa to 145 MPa at mid span and then decreases again. The minimum hoop stress occurs at outer radius and it is equal to 114 MPa. This distribution resembles the axi-symmetric case with lamination angle of \((\pm 30^\circ)\), shown in Fig. (3.26).

Figure (3.29) shows that the in-plane shear stresses are of small values. Figure (3.30) shows that the radial displacements around the constrained nodes are of small values as compared to the other side, where maximum displacements occur. The boundary
conditions used are the cause of the ovality of the inner radius deformed shape. The actual displacement at the outer surface can be calculated as the average of the displacements of the two points shown on the figure.

Figure (3.31) shows the hoop displacement to be symmetric across the radial line, where the constrained nodes are located. Theoretically, the hoop displacement is independent of $\theta$, however due to the applied boundary conditions, the hoop displacement is non-uniform with respect to $\theta$. The symmetry indicates the existence of hoop displacement and its uniformity, irrespective of the apparent change of value.

Figure (3.32) shows the slimming effect of the disk. To maintain a constant volume, the radial expansion is compensated by a decrease in length. The two free surfaces of the disk move inward with equal distances.
3.5 PROPOSED DESIGN PARAMETERS FOR THE FMW ROTOR

Based on the results presented in the previous sections of this chapter, a suggested design is presented for the FMW rotor that would be coupled with the magnetic bearings to fulfill the 55 Nms angular momentum storage requirement. The isotropic rotor solution using aluminum is discarded because of its limited performance at high speeds. The solution of the axi-nonsymmetric case is also discarded because it does not fit the application in hand, it should rather be used in an application where external moments are considered as a loading condition. In this case, using a lamination angle which can provide material characteristics that produce reaction torques equal and opposite to those applied on the body is recommended. The optimum solution for the application of FMW rotor is the axi-symmetric laminated rotor.

The suggested rotor design consists of 2 concentric rings of an inner radius of 45 mm and an outer radius of 150 mm. The inner ring is made out of steel and the outer one is made out of 70% unidirectionally-graphite-reinforced-epoxy laminate. The reinforcement is oriented in the hoop direction, this is because although the hoop stresses can be decreased by using a $(\pm 20^\circ)$, axi-symmetric laminated rotor, the accompanying increase in the radial stresses will decrease the factor of safety of the part. The contact radius between the steel ring and the graphite-reinforced-epoxy ring is 50 mm, and the amount of interference is 0.01 mm. For a total mass of the rotor of 1.2 kg, and for an output maximum momentum of 55 Nms, the rotor length is 9.275 mm and the maximum angular velocity is 42,735 rpm. According to Tsai-Wu failure criterion, the factor of safety associated with those design parameters is 1.33. At this maximum operational speed, the radial and hoop stress distributions and the radial displacement are shown in Fig. (3.33), (3.34) and (3.35) respectively.
Fig. (3.33) Radial Stress Distribution for Optimum Solution @ 42,735 rpm

Fig. (3.34) Hoop Stress Distribution for Optimum Solution @ 42,735 rpm
Fig. (3.35) Radial Displacement for Optimum Solution @ 42,735 rpm
Chapter Four

Conclusions

The closed form mathematical models developed for the isotropic as well as the axisymmetric laminated cases and the finite element model developed for the axi-nonsymmetric laminated case proved that the performance of conventional isotropic steel-aluminum rotors is well-below that of the suggested GRE laminated rotors in terms of stiffness and strengths. Furthermore, the study shows that the behavior of axi-symmetric GRE laminate is different than that of isotropic materials under constant angular velocity as the only loading condition. For isotropic materials, the hoop stress is maximum at the rotor inner radius; whereas for axi-symmetric laminated material, with the fibers aligned in the hoop direction, the hoop stress is maximum at a point close to the outer radius. It was also proved that by changing the lamination angle from 0° to 40°, a case close to the isotropic behavior, the hoop stress changed to a distribution close to that of the isotropic case. The finite element analysis shows a similar behavior for the axi-nonsymmetric laminate.

The parametric analysis carried out on the axi-symmetric laminated rotor design proved that it is more efficient to use minimum interferences between the neighboring rings, just to keep them in contact while rotating at maximum operating speed. The analysis also proved that there is an inverse relation between the reinforcement volume fraction and the amount of interferences needed to keep the rings in contact while operating at same speeds. It was found out that the minimum interferences needed for a 30% GRE rotor with 0° lamination angle are double those needed for a 60% GRE rotor with 0° lamination angle.

The parametric analysis of axi-symmetric laminated GRE rotors also proves that for such an application, operating at high speeds, the hoop stresses are the dominant factor for failure, therefore it is advised to orient all the fibers in the hoop direction to provide stiffness and strength in this direction. Although the study of the behavior of axi-symmetric laminate shows that the hoop stresses are minimal for a lamination angle of (±20°), the associated maximum radial stress in this case increases to more than four times those of the 0° lamination angle. Consequently, the maximum factor of safety can be obtained by using a 0° lamination angle for the design of the GRE rotor. Finally, it was found out that for such an
application, it is more advantageous to use minimum number of rings to minimize maximum hoop stresses. This is because although the interference decreases the hoop stresses of the inner ring, it increases the hoop stresses of the outer ring, which for a reinforced laminate exhibits higher stresses naturally.

From the analysis carried out on the axi-nonsymmetric laminated rotor design, it was found out that there exists a complicated closed form solution for the stresses, strain and displacement, which are not only function of the radius, but also of the length of the rotor. Therefore, this problem was solved using the Finite Element Method.

Finally, a 1.2 kg axi-symmetric laminated rotor running at a speed of 42,735 rpm is suggested for the design of the magnetically suspended FMW rotor. The rotor consists of two concentric rings of inner and outer radii of 45 and 150 mm respectively. The inner ring is made out of steel to provide the required interaction with the magnetic bearings and is only 5 mm thick, while the outer ring is made out of 70% graphite-reinforced-epoxy with 0° lamination angle. The amount of interference needed is 0.01 mm. Based on the thermal analysis, the outer ring needs to be heated to a temperature of 27°C to create the required interference. The factor of safety involved for the design is 1.33. This reduction in weight from 4 kg to 1.2 kg accounts for a reduction of at least 168,000 US $ of launch costs.
References (List 1)


References (List 2)

2.1- Serry, M., The Egyptian Satellite Co. Nilesat, Private Communication, 1999
2.3- Matra Marconi Space, Nilesat Operations, Cosmic Training Course, 1998
2.5- Boissiere, Ph., Matra Marconi Space, Private Communication, 1998
2.6- Teldix homepage: www.teldix.com/p22.htm product & services: space products
Appendix A
Teldix FMW Performance Summary

Listed below are the performance characteristics of Teldix Fixed Momentum wheel [2,6].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Momentum storage capacity (5060 rpm)</td>
<td>55 Nms</td>
</tr>
<tr>
<td>Rotor inertia</td>
<td>0.1038 m²Kg</td>
</tr>
<tr>
<td>Nominal wheel speed (50 Nms)</td>
<td>4600 rpm</td>
</tr>
<tr>
<td>Friction torque</td>
<td>&lt; 0.02 Nm at 5060 rpm</td>
</tr>
<tr>
<td>Motor scale factor</td>
<td>0.073 Nm/A</td>
</tr>
<tr>
<td>Max reaction torque</td>
<td>0.1 Nm</td>
</tr>
<tr>
<td>Mass</td>
<td>7.5 kg</td>
</tr>
<tr>
<td>Dimensions</td>
<td>Diameter: 343 mm</td>
</tr>
<tr>
<td>Wheel spin up time to nominal speed</td>
<td>H: 120 mm</td>
</tr>
<tr>
<td>Mechanical stability of momentum vector</td>
<td>500 sec</td>
</tr>
<tr>
<td>Misalignment between momentum vector and mounting plane</td>
<td>&lt; 0.6 arcmin</td>
</tr>
<tr>
<td>Momentum vector direction</td>
<td>&lt; 1 arcmin (half cone)</td>
</tr>
<tr>
<td>Misalignment measurement knowledge</td>
<td>&lt; 3 arcmin (half cone)</td>
</tr>
<tr>
<td>Dynamic unbalance</td>
<td>&lt; 2 arcmin</td>
</tr>
<tr>
<td>Static unbalance</td>
<td>&lt; 30 microseconds</td>
</tr>
<tr>
<td>Voltage supply:</td>
<td>&lt; 26 micros</td>
</tr>
<tr>
<td>motor</td>
<td>0 to 45 V</td>
</tr>
<tr>
<td>commutation electronics</td>
<td>+15V ± 10%</td>
</tr>
<tr>
<td>Power consumption:</td>
<td>&lt; 0.4 W</td>
</tr>
<tr>
<td>commutation electronics</td>
<td>&lt; 12 W</td>
</tr>
<tr>
<td>motor steady state consumption at 5060 rpm</td>
<td>&lt; 83 W</td>
</tr>
<tr>
<td>motor max output torque consumption at 5060 rpm</td>
<td></td>
</tr>
</tbody>
</table>
Appendix B

Honeywell HM4520 Momentum Wheel Performance Summary

Listed below are all the performance characteristics of Honeywell Momentum wheel [2.8]

**Standard Performance Characteristics**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Capability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Momentum</td>
<td>N-m·s</td>
<td>90/95</td>
</tr>
<tr>
<td>Output Torque</td>
<td>N·m</td>
<td>0.135</td>
</tr>
<tr>
<td>Peak Power</td>
<td>Watts</td>
<td>39</td>
</tr>
<tr>
<td>Power Holding</td>
<td>Watts</td>
<td>35</td>
</tr>
<tr>
<td>Max Speed</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Power Bus Voltage</td>
<td>Volts</td>
<td>51</td>
</tr>
<tr>
<td>Wheel Speed</td>
<td>rpm</td>
<td>±5400</td>
</tr>
<tr>
<td>Weight</td>
<td>Kg</td>
<td>2.55</td>
</tr>
<tr>
<td>Outside Diameter</td>
<td>mm</td>
<td>406</td>
</tr>
<tr>
<td>Inside Diameter</td>
<td>in.</td>
<td>16</td>
</tr>
<tr>
<td>Height</td>
<td>mm</td>
<td>215</td>
</tr>
<tr>
<td>Height</td>
<td>in.</td>
<td>8.5</td>
</tr>
<tr>
<td>Integral Electronics</td>
<td>Yes/No</td>
<td>Yes</td>
</tr>
<tr>
<td>Life Requirement</td>
<td>Yes</td>
<td>15</td>
</tr>
<tr>
<td>Radiation Hard</td>
<td>Yes/No</td>
<td>Yes</td>
</tr>
<tr>
<td>Part Screening</td>
<td>Level</td>
<td>5</td>
</tr>
<tr>
<td>Bearing</td>
<td>Size</td>
<td>101</td>
</tr>
<tr>
<td>Operational (Qual)</td>
<td>deg·sec</td>
<td>40·sec</td>
</tr>
<tr>
<td>Temp (Amb)</td>
<td>deg·C</td>
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</tr>
<tr>
<td>Vibration (Qual)</td>
<td>Grms</td>
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<tr>
<td>Micromag</td>
<td>A/m</td>
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</tr>
<tr>
<td>Interface</td>
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<td>Analog/Digital</td>
</tr>
<tr>
<td>Static Unbalance</td>
<td>oz·in.</td>
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</tr>
<tr>
<td>Dynamic Unbalance</td>
<td>gm·cm²</td>
<td>36.4</td>
</tr>
</tbody>
</table>
Appendix C

Teldix and Honeywell Reaction Wheels

Shown below in Fig. (A-1) Teldix Reaction wheel used for Low Earth Orbit satellites [2.6]. Also Honeywell Reaction Wheel HR 2020 is shown in Fig. (A.2) and all its performance characteristics are listed [2.8].

Fig. (A.1) Teldix LEO Satellites Reaction Wheel [2.6]

*Model HR 2020 Crosssection View*

Fig. (A.2) Honeywell HR 2020 Reaction Wheel [2.8]
## Standard Performance Characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Capability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Momentum</td>
<td>N-m</td>
<td>20</td>
</tr>
<tr>
<td>Output Torque</td>
<td>oz-in.</td>
<td>0.113</td>
</tr>
<tr>
<td>Peak Power</td>
<td>Watts</td>
<td>175</td>
</tr>
<tr>
<td>Power Holding Max Speed</td>
<td>Watts</td>
<td>35</td>
</tr>
<tr>
<td>Power Bus Voltage</td>
<td>Watts</td>
<td>70</td>
</tr>
<tr>
<td>Wheel Speed</td>
<td>rpm</td>
<td>6500</td>
</tr>
<tr>
<td>Weight</td>
<td>kg</td>
<td>7.9</td>
</tr>
<tr>
<td>Outside Diameter</td>
<td>in.</td>
<td>11.8</td>
</tr>
<tr>
<td>Height</td>
<td>in.</td>
<td>17.2</td>
</tr>
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<td>Yes/No</td>
<td>Yes</td>
</tr>
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<td>Yes</td>
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<tr>
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<tr>
<td>Bearing</td>
<td>Size</td>
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</tr>
<tr>
<td>Operational (Qual) Temp Range</td>
<td>deg C Low</td>
<td>172</td>
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<tr>
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<td>Grms</td>
<td>13.3</td>
</tr>
<tr>
<td>Motor Type</td>
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<td>Interface</td>
<td>Analog/Digital</td>
<td>Digital</td>
</tr>
<tr>
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<tr>
<td>Dynamic Unbalance</td>
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<td>0.50</td>
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Appendix D

Analysis of Elastic Stress & Strain in Polar Coordinates

The stress equations are developed in the theory of elasticity by enforcing the equilibrium status on the surface tractors as well as the body forces of an infinitesimal element. Let $\sigma_{ij}, \ i,j = \theta, z, r$ be the stress tensor, and $F_i, \ i = \theta, z, r$ be body forces per unit volume. The resulting three equilibrium (stress-displacement) equations are [1.24]:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{r\theta} - \sigma_{\theta r}}{r} + F_r = 0
\]  
(D.1)

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \tau_{\theta \theta}}{\partial \theta} + \frac{\tau_{\theta \theta}}{r} + F_\theta = 0
\]  
(D.2)

\[
\frac{\partial \tau_{\theta z}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\tau_{\theta z}}{r} + F_z = 0
\]  
(D.3)

The strain equations are developed in the theory of elasticity by geometrical relations between strains and displacements. Let $\varepsilon_i, \ i = \theta, z, r$ denote normal strain tensor and $\gamma_{ij}, \ i,j = \theta, z, r$ denote the shearing strain tensor. Also let $v, w, u$, denote the hoop, axial, and radial displacements respectively. The resulting six strain-displacement equations are [1.24]:

\[
\varepsilon_r = \frac{\partial u}{\partial r}
\]  
(D.4)

\[
\varepsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}
\]  
(D.5)

\[
\varepsilon_z = \frac{\partial w}{\partial z}
\]  
(D.6)

\[
\gamma_{\theta \theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} + \frac{v}{r}
\]  
(D.7)

\[
\gamma_{\theta z} = \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}
\]  
(D.8)

\[
\gamma_{zz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}
\]  
(D.9)
The stress-strain relations in polar coordinates are expressed by the generalized Hook's law as follows [1.24]:

$$\sigma = [L] \varepsilon$$  \hspace{1cm} (D.10)

$$\varepsilon = [M] \sigma$$  \hspace{1cm} (D.11)

where $[L]$ is called the stiffness matrix and $[M]$ is the compliance matrix, and $[M] = [L]^t$. In a matrix notation, it can be rewritten as [1.24]:

$$\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{bmatrix} = [L]
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix}$$  \hspace{1cm} (D.12)

where, for polar coordinates: $\theta = 1$, $z = 2$, and $r = 3$. The shape of the stiffness matrix depends on the degree of isotropy of the material. This is discussed in section (2.1).
Appendix E

Proof of Cauchy-Euler Differential Operators Relationships

First Relationship
\[ rD\mu = D_r\mu \]

First Proof
\[ D_r\mu = \frac{du}{dr} = \frac{du}{dt} \frac{dr}{dt} = \frac{du}{dr} = rD\mu \]

Second Relationship
\[ r^2 D^2 \mu = D_r(D_r - 1)\mu \]

Second Proof
\[ D_r(D_r - 1)\mu = D_r^2 - D_r\mu \]
\[ = \frac{d^2 \mu}{dt^2} - \frac{du}{dt} \frac{du}{dr} \]
\[ = \frac{d}{dt} \left( \frac{du}{dr} \right) = \frac{du}{dt} \]
\[ = \frac{d}{dr} \left( r \frac{du}{dr} \right) - r \frac{du}{dr} \]
\[ = \left( r \frac{d^2 \mu}{dr dt} + \frac{du}{dr} \frac{dr}{dt} \right) = r \frac{du}{dr} \]
\[ = \left( r \frac{d^2 \mu}{dr dt} + \frac{du}{dr} \frac{dr}{dt} \right) = r \frac{du}{dr} \]
\[ = r^2 D^2 \mu \]
\[ + r \frac{du}{dr} - r \frac{du}{dr} \]
\[ = r^2 D^2 \mu \]

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Appendix F

Matlab Program for Displacement and Stresses Calculation for 1 Steel Ring, 5 Axi-symmetric GRF Rings Rotor

% Inner Steel Ply Calculations
% Steel Properties
Es = 207e9; % kg/m.s²
NuS = 0.3;
% Steel Young's Modulus
Ls = (Es/(1-NuS²)) * [1 NuS 0; NuS 1 0; 0 0 1-NuS²];
% Steel Poisson's Ratio

% Matrix Properties
Em = 2.5e9; % kg.m.s²
Num = 0.4;
Gm = Em/(2*(1+Num));
Kbm = Em/(3*(1-2*Num));
% Epoxy Young's Modulus
% Epoxy Poisson's Ratio
% Epoxy Shear Modulus
% Epoxy Bulk's Modulus

% Matrix Hill's Modulii
km = Kbm+Gm/3;
lm = Kbm-2*Gm/3;
nm = Kbm+4*Gm/3;
pm = Gm;
mn = Gm;
% Graphite Axial Young's Modulus
% Graphite Transverse Young's Modulus
% Graphite Symmetric Plane Poisson's Ratio
% Graphite Transverse Plane Poisson's Ratio
% Graphite Shear Modulus
% Graphite Shear Modulus

Mf = [1/Ef; -NuFA/EfA; -NuFA/EfA; 0 0 0; -NuFA/EfA; 1/EfT; -NuT/EfT; 0 0 0; -NuFA/EfA; -NuT/EfT; 1/EfT; 0 0 0 0 0 0 0 0 0 1/GfA; 0 0 0 0 0 0 0 0 0 1/GfA];
Lff = inv(Mf);
% Fibers Hill's Modulii
kf = Lff(2,2) - Lff(4,4);
lf = Lff(1,2);
f = Lff(1,1);
pf = Lff(3,5);
mf = Lff(4,4);
% Composite Ring's Local & Global Properties Calculations
% This varies only based on the reinforcement volume fraction & angle of each ply.
% Composite Ring (1) Calculations
% Fiber & Matrix Volume Fractions for Ring (1)
Cf1 = input('Cf1 = ');% The Overall Hills Modulii For Ring (1)
p1 = (2*Cf1*pm*pf+Cm1*(pm*pf+pm²))/2*Cf1*(pm+Cm1*(pf+pm));
ml = (nm²*mf*(km+2*nm)+km*nm²*(Cf1*mf+Cm1*nm))/(km*nm*(km+2*nm)*(Cf1+nm+Cm1*mf));
k1 = (Cf1*km*(km+mm)+Cm1*km²*(km+mm))/(Cf1*(km+mm)+Cm1*(km+mm));
l1 = (Cf1*km*(km+mm)+Cm1*km²*(km+mm))/(Cf1*(km+mm)+Cm1*km²*(km+mm));
n1 = Cm1 *nm+Cf1 *nF+(11-CF1 *IF-Cm1+I'm) * (IF-I'm)/(kF-km);
%

% The Local Stiffness and Compliance Matrices For Ply (1): Transversely Isotropic
L1 = [n1 11 1 0 0 0; 1 (k1+m1) (k1-m1) 0 0 0 0; 1 (k1-m1) (k1+m1) 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0];
M1 = inv(L1);
%
% Plane Stress 3x3 Local Compliancie & Stiffness Matrices For Ring (1)
Ms1 = [M1(1,1) M1(1,3) 0; M1(3,1) M1(3,3) 0.0 0 M1(3,5)];
Ls1 = inv(Ls1);
%
% Transformation of Local Compliance & Stiffness Matrices into Global Coordinates For Ring(1)
% the positive rotation
ang11 = input('ang11=');
% in degrees
angr11 = ang11 * 2 * 22/7 / 360;
% in radians
a11 = cos(angr11);
b11 = sin(angr11);
T11 = [a11^2, b11^2, -2*a11*b11; b11^2, a11^2, -2*a11*b11; -a11*b11, a11*b11, a11^2+b11^2];
Mg11 = T11 * Ms1 * T11;
Lg11 = inv(Mg11);
% The negative rotation
ang12 = - ang11;
% in degrees
angr12 = ang12 * 2 * 22/7 / 360;
% in radians
a12 = cos(angr12);
b12 = sin(angr12);
T12 = [a12^2, b12^2, -2*a12*b12; b12^2, a12^2, -2*a12*b12; -a12*b12, a12*b12, a12^2+b12^2];
Mg12 = T12 * Ms1 * T12;
Lg12 = inv(Mg12);
% Overall Stiffness
Lg1 = 0.5 * (Lg11 + Lg12);
%
% Factors Used in Calculations for Ring (1)
s2 = Lg1(1,1);
t2 = Lg1(1,2);
t2 = Lg1(1,2);
h2 = Lg1(1,3);
%
% Composite Ring (2) Calculations
% Fiber & Matrix Volume Fractions For ring (2)
CF2 = input ('CF2=');
Cm2 = 1-CF2;
%
% The Overall Hills Moduli For Ring (2)
p2 = (2*CF2*pm*pf+Cm2*(pm*pf+pm^2))/(2*CF2*pm+Cm2*pm+(pf+pm^2));
m2 = (mm*nm^2+(km+2*mm)^2+km*mm*(CF2*nm+Cm2+mm)) / (km+2*mm+mm*CF2*mm+Cm2*mm);
k2 = (CF2*kF*(km+mm)+Cm2*kF*km) / (CF2*km+mm+Cm2*km);
l2 = (CF2*kF*(km+mm)+Cm2*mm*(km+mm)) / (CF2*km+mm+Cm2*km);
%(CF2*km+mm+Cm2*km);
%
% The Local Stiffness and Compliance Matrices For Ring (2); Transversely Isotropic
L2 = [n2 12 0 0 0 0; 12 (k2+m2) (k2-m2) 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0];
M2 = inv(L2);
%

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% Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (2)
\[ M_{s2} = \begin{bmatrix} M_{2}(1,1) & M_{2}(1,3) & 0; & M_{2}(3,1) & M_{2}(3,3) & 0.0 & 0 & M_{2}(5,5) \end{bmatrix}; \]
\[ L_{s2} = \text{inv}(L_{s2}); \]
% % Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (2)
% The positive rotation
% \[ \text{ang}21 = \text{input}(\text{\textquoteleft}\text{ang}2=\text{\textquoteleft}); \% \text{in degrees} \]
% \[ \text{angr}21 = \text{ang}21*2*227/360; \% \text{in radians} \]
% \[ a21 = \cos(\text{angr}21); \]
% \[ b21 = \sin(\text{angr}21); \]
% \[ T_{21} = [a21^21^2, b21^21^2, 2*a21*b21, b21*a21, a21^2, b21^2, -2*a21*b21, -a21*b21, a21^2-b21^2]; \]
% \[ M_{g21} = T_{21}^*M_{s2}^*T_{21}; \]
% \[ L_{g21} = \text{inv}(M_{g21}); \]
% The negative rotation
% \[ \text{ang}22 = -\text{ang}21; \% \text{in degrees} \]
% \[ \text{angr}22 = \text{ang}22*2*227/360; \% \text{in radians} \]
% \[ a22 = \cos(\text{angr}22); \]
% \[ b22 = \sin(\text{angr}22); \]
% \[ T_{22} = [a22^22^2, b22^22^2, 2*a22*b22, b22*a22, a22^2, b22^2, -2*a22*b22, -a22*b22, a22^2-b22^2]; \]
% \[ M_{g22} = T_{22}^*M_{s2}^*T_{22}; \]
% \[ L_{g22} = \text{inv}(M_{g22}); \]
% The overall stiffness
% \[ L_{g2} = 0.5*(L_{g12}+L_{g22}); \]
% % Factors Used in Calculations for Ring(2)
% \[ s_3 = L_{g2}(1,1); \]
% \[ t_3 = L_{g2}(1,2); \]
% \[ z_3 = L_{g2}(2,2); \]
% \[ h_3 = L_{g2}(3,3); \]
% % % Composite Ring (3) Calculations
% Fiber & Matrix Volume Fractions For Ring (3)
% \[ C_{f3} = \text{input}(\text{\textquoteleft}C_{f3}=\text{\textquoteleft}); \]
% \[ C_{m3} = 1-C_{f3}; \]
% % The Overall Hills Moduli For Ring (3)
% \[ p_3 = (2*C_{f3}^2+C_{m3}^2+C_{f3}^2*C_{m3}^2)/(2*C_{f3}^2-C_{m3}^2); \]
% \[ m_3 = (C_{f3}^2+C_{m3}^2)/(C_{f3}^2+C_{m3}^2); \]
% \[ k_3 = (C_{f3}^2*C_{m3}^2)/(C_{f3}^2+C_{m3}^2); \]
% \[ l_3 = (C_{f3}^2*C_{m3}^2)/(C_{f3}^2+C_{m3}^2); \]
% \[ n_3 = (C_{f3}^2*C_{m3}^2)/(C_{f3}^2+C_{m3}^2); \]
% % The Local Stiffeness and Compliance Matrices For Ply (3): Transversely Isotropic
% \[ L_3 = [n_3 l_3 k_3 m_3 l_3 k_3 m_3 k_3 m_3 n_3 1 1 1; 1 1 1 1 1 1 1 1 1]; \]
% \[ L_{33} = \text{inv}(L_3); \]
% % Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (3)
% \[ L_{s3} = \text{inv}(L_{s3}); \]
% % % Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (3)
% The positive rotation
% \[ \text{ang}31 = \text{input}(\text{\textquoteleft}\text{ang}3=\text{\textquoteleft}); \% \text{in degrees} \]
% \[ \text{angr}31 = \text{ang}31*2*227/360; \% \text{in radians} \]
\( a_{31} = \cos(\text{angr31}) \);
\( b_{31} = \sin(\text{angr31}) \);
\( T_{31} = [a_{31}]^2 \cdot b_{31}^2, 2 \cdot a_{31} \cdot b_{31}; b_{31}^2, a_{31}^2, -2 \cdot a_{31} \cdot b_{31}; -a_{31}^2 \cdot b_{31}, a_{31} \cdot b_{31}; a_{31}^2 \cdot b_{31}^2 ] \);
\( M_{31} = T_{31} \cdot M_{33}^* \cdot T_{31}^* \);
\( L_{31} = \text{inv}(M_{31}) \);
% The negative rotation
\( \text{angr32} = -\text{angr31} \);
\( \text{angr32} = \text{angr32} \cdot 2 \cdot 227/7 \cdot 360 \);
% in degrees
\( a_{32} = \cos(\text{angr32}) \);
\( b_{32} = \sin(\text{angr32}) \);
\( T_{32} = [a_{32}]^2 \cdot b_{32}^2, 2 \cdot a_{32} \cdot b_{32}; b_{32}^2, a_{32}^2, -2 \cdot a_{32} \cdot b_{32}; -a_{32}^2 \cdot b_{32}, a_{32} \cdot b_{32}, a_{32}^2 \cdot b_{32}^2 ] \);
\( M_{32} = T_{32} \cdot M_{33}^* \cdot T_{32}^* \);
\( L_{32} = \text{inv}(M_{32}) \);
% the overall stiffness
\( L_{3} = 0.5 \cdot (L_{31} + L_{32}) \);
% Factors Used in Calculations for Ring (3)
\( s_{4} = L_{3} \cdot L_{3}(1,1) \);
\( t_{4} = L_{3} \cdot L_{3}(1,2) \);
\( z_{4} = L_{3} \cdot L_{3}(2,2) \);
\( h_{4} = L_{3} \cdot L_{3}(3,3) \);
% %
% Composite Ring (4) Calculations
% Fiber & Matrix Volume Fractions For Ring (4)
\( C_{f4} = \text{input} \left( \text{Cf4} = \cdot \right) \);
\( C_{m4} = 1 - C_{f4} \);
% The Overall Hills Moduli For Ring (4)
\( p_{4} = \frac{(2 \cdot C_{f4} \cdot p_{m} + p_{f} \cdot C_{m4} \cdot p_{m} + C_{m4} + p_{f} \cdot p_{m})}{(2 \cdot C_{f4} \cdot p_{m} + C_{m4} + p_{f} \cdot p_{m})} \);
\( m_{4} = \frac{(m_{n} \cdot n_{f} \cdot (m_{n} + 2 \cdot n_{m}) \cdot (m_{n} \cdot n_{m} \cdot C_{f4} + m_{n} \cdot n_{m} \cdot C_{m4} + m_{f} \cdot n_{f}))}{(m_{n} \cdot n_{f} \cdot (m_{n} + 2 \cdot n_{m}) \cdot (m_{n} \cdot n_{m} \cdot C_{f4} + m_{n} \cdot n_{m} \cdot C_{m4} + m_{f} \cdot n_{f}))} \);
\( k_{4} = \frac{(C_{f4} \cdot k_{m} \cdot (m_{m} + n_{m}) \cdot C_{m4} + k_{m} \cdot (m_{m} + n_{m}) \cdot C_{m4} + (k_{f} \cdot n_{f}))}{(C_{f4} \cdot k_{m} \cdot (m_{m} + n_{m}) \cdot C_{m4} + k_{m} \cdot (m_{m} + n_{m}) \cdot C_{m4} + (k_{f} \cdot n_{f}))} \);
\( l_{4} = \frac{(C_{f4} \cdot l_{f} \cdot (k_{m} + n_{m}) \cdot C_{m4} + l_{f} \cdot (k_{m} + n_{m}) \cdot C_{m4} + (l_{m} \cdot k_{m}))}{(C_{f4} \cdot l_{f} \cdot (k_{m} + n_{m}) \cdot C_{m4} + l_{f} \cdot (k_{m} + n_{m}) \cdot C_{m4} + (l_{m} \cdot k_{m}))} \);
\( n_{4} = \frac{(C_{m4} \cdot n_{m} + C_{f4} \cdot n_{f} + (l_{1} - C_{f4} \cdot l_{f} \cdot C_{m4} + l_{m} \cdot (l_{m} - k_{m}))}{(l_{f} - l_{m}) \cdot (k_{f} - k_{m})} \);
% The Local Stiffness and Compliance Matrices For Ring (4); Transvertely Isotropic
\( L_{4} = \begin{bmatrix} 0.44 & 0.40 & 0.00 & 0.00 & 0.40 & 0.40 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix} \);
\( M_{4} = \text{inv}(L_{4}) \);
% Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (4)
\( M_{4} = [M(4,1), M(4,1), 0; M(4,3), M(4,3), 0; 0, 0, M(4,5)] \);
\( L_{4} = \text{inv}(L_{4}) \);
% Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (4)
% Positive rotation
\( \text{angr4} = \text{input} \left( \text{angr4} = \cdot \right) \);
\( \text{angr4} = \text{angr4} \cdot 2 \cdot 227/7 \cdot 360 \);
% in degrees
\( a_{41} = \cos(\text{angr4}) \);
\( b_{41} = \sin(\text{angr4}) \);
\( T_{41} = [a_{41}]^2 \cdot b_{41}^2, 2 \cdot a_{41} \cdot b_{41}; b_{41}, a_{41}, -2 \cdot a_{41} \cdot b_{41}; -a_{41}^2 \cdot b_{41}, a_{41} \cdot b_{41}, a_{41}^2 \cdot b_{41}^2 ] \);
\( M_{41} = T_{41}^* \cdot M_{4} \cdot T_{41}^* \);
\( L_{41} = \text{inv}(M_{41}) \);
% The negative rotation
\( \text{angr4} = \text{angr4} \);
\( \text{angr4} = \text{angr4} \cdot 2 \cdot 227/7 \cdot 360 \);
% in degrees
\( a_{42} = \cos(\text{angr4}) \);
\( b_{42} = \sin(\text{angr4}) \);

\text{end}
a42=cos(angr42);
b42=sin(angr42);
T422=[a42^2, b42^2, 2*a42*b42; b42^2, a42^2, -2*a42*b42; -a42*b42, a42*b42, a42^2-b42^2];
Mg42 = T422*Mg4*Mg42;
Lg42 = inv(Mg42);
% The overall stiffness
Lg4 = 0.5*(Lg41+Lg42);
%
% Factors Used in Calculations for Ring (4)
s5 = Lg4(1,1);
t5 = Lg4(1,2);
z5 = Lg4(2,2);
h5 = Lg4(3,3);
%
%
% Ply (5) Calculations
% Fiber & Matrix Volume Fractions For Ring (5)
Cf5 = input ('Cf5= ');
Cm5 = 1-Cf5;
%
% The Overall Hills Moduli For Ring (5)
p5 = (2*Cf5*pm*pf+Cm5*(pm*pf-pm)^2)/(2*Cf5*pm+Cm5*(pf+pm));
m5 = (km*mf)+(km+2*mm)*(Cf5*mf+Cm5*nm)/(km*mf+(km+2*mm)*(Cf5*nm+Cm5*mf));
k5 = (Cf5*km+(km+mm)*Cm5*km)/(Cf5*(km+mm)+Cm5*(km+mm));
l5 = (Cf5*km+(km+mm)*Cm5*lm)/(Cf5*(km+mm)+Cm5*(km+mm));
a5 = Cm5*nm+Cf5*nf+(1-Cf5*lf*Cm5*lm)*(l1-lm)/(kl-km);
%
% The Local Stiffness and Compliance Matrices For Ring (5); Transversely Isotropic
L5 = [n5 l5 0 0 0; l5 (k5+m5) (k5-m5) 0 0 0; 0 0 0 m5 0 0; 0 0 0 0 m5 0 0; 0 0 0 p5 0 0 0 0];
M5 = inv(L5);
%
% Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (5)
Ms5 = [M5(1,1) M5(1,3) 0; M5(3,1) M5(3,3) 0; 0 0 0 M5(5,5)];
Ls5 = inv(Ls5);
%
% Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (5)
% The positive rotation
ang51 = input ('ang51= ');  % in degrees
angr51 = ang51*2*22/7/360;  % in radians
a51=cos(angr51);
b51=sin(angr51);
T51=[a51^2, b51^2, 2*a51*b51; b51^2, a51^2, -2*a51*b51; -a51*b51, a51*b51, a51^2-b51^2];
Mg51 = T51*Ms5*T51;
Lg51 = inv(Mg51);
% the negative rotation
ang52 = -ang51;  % in degrees
angr52 = ang52*2*22/7/360;  % in radians
a52=cos(angr52);
b52=sin(angr52);
T52=[a52^2, b52^2, 2*a52*b52; b52^2, a52^2, -2*a52*b52; -a52*b52, a52*b52, a52^2-b52^2];
Mg52 = T52*Ms5*T52;
Lg52 = inv(Mg52);
% The overall stiffness
Lg5 = 0.5*(Lg51+Lg52);
% Factors Used in Calculations for Ring (5)
\[ s_6 = \log(1.1) \]
\[ i_6 = \log(1.2) \]
\[ z_6 = 1 - 0.5(2.2) \]
\[ h_6 = \log(3.3) \]
%
% Solving the Differential Equations
%
% Calculating the Each Ply Density, Overall Density, Overall Mass, Angular Momentum
ROHs = 7800; % Kg/m^3
ROHm = 1810; % Kg/m^3
ROHf = 1900; % Kg/m^3
Ri = 0.045; % m
Rc1 = 0.05; % m
Rc2 = 0.07; % m
Rc3 = 0.09; % m
Rc4 = 0.11; % m
Rc5 = 0.13; % m
R0 = 0.15; % m
ROH1 = ROHs;
ROH2 = (C1*ROHf+Cm1*ROHm);
ROH3 = (C2*ROHf+Cm2*ROHm);
ROH4 = (C3*ROHf+Cm3*ROHm);
ROH5 = (C4*ROHf+Cm4*ROHm);
ROH6 = (C5*ROHf+Cm5*ROHm);

Mass = input ('Mass=')

Depth = Mass/((22/7) * (ROH1*(Rc1^2-Rc2^2) + ROH2*(Rc2^2-Rc1^2) + ROH3*(Rc3^2-Rc2^2) + ROH4*(Rc4^2-Rc3^2) + ROH5*(Rc5^2-Rc4^2) + ROH6*(Rc6^2-Rc5^2))) % in m
iz = 0.5*Mass/Ro; % in Kg,m^2
omega_rps = 55/iz; % in rad/sec. H = 55 Nm/S
omega = omega_rps * 60/(2*22/7) % in rpm
%
% Design Parameter (interfaces) Input
delta1 = input ('Delta1='); % Amount of Interference at Contact Surface 1 in m
delta2 = input ('Delta2='); % Amount of Interference at Contact Surface 2 in m
delta3 = input ('Delta3='); % Amount of Interference at Contact Surface 3 in m
delta4 = input ('Delta4='); % Amount of Interference at Contact Surface 4 in m
delta5 = input ('Delta5='); % Amount of Interference at Contact Surface 5 in m
%
% Solving for the Constants of integration of the displacement and stresses functions:
% eq. 2 SigmaR2 - SigmaR1 = 0 @ Rc1 (The coeff's A&C of the stc ring are put as % functions of C1 & C2. That's done by using two equations: eq. 1 SigmaR1 @ Rc1 = 0 %
% & eq. 3 delta1 = u2-u1 @ Rc1.
q1 = ((1/Rc1^2 - 1/Rc2^2)*(Rc1^2*sqrt(s2^2))*((1+Nus)/(Es*Rc1) + (1-Nus)*Rc1/(Es*Rc1)^2) + (12+sqrt(s2^2))*Rc1/(sqrt(s2^2) -1));
q2 = (((1/Rc1^2 - 1/Rc2^2)/(Rc1^2*sqrt(s2^2)))/(1+Nus)/((Es*Rc1) + (1-Nus)*Rc1/(Es*Rc1)^2) + (12-sqrt(s2^2))*Rc1/(sqrt(s2^2) +1));
line2 = [q2, 0, 0, 0, 0, q1, .0, 0, 0, 0, 0];
equal2 = (3*z2^2)*ROH2*(omega_rps*Rex^2)*2)*(8*z2^2) +
(Rex^2 - Rex^2)*14+(3+Nus)*ROH1*(omega_rps^2)/8 +
(1/Rc1^2 - 1/Rc2^2)*delta1 + ROH2*(omega_rps^2)*Rc1^3/(8*Es) +
(Nus-2)*ROH1*(omega_rps^2)*Rc1^3/(8*Es) +
(1-Nus)*(3+Nus)*ROH1*(omega_rps^2)*Rc1^3/((1+Nus)/(Es*Rc1)) +
(1-Nus)*Rc1/(Es*Rc1)^2);
% eq. 4 SigmaR3 - SigmaR2 = 0 @ Rc2
line4 = ((sqrt(s2^2+2^2)/(Rc2^2-sqrt(s2^2)+1)), (3-sqrt(s3^3+3))/(Rc2^2-sqrt(s3^3)+1)), 0.0, 0.0.
line5 = (1/-Rc2^2*(sqrt(s2^2)+1)), 1/(Rc2^2-sqrt(s3^3)+1)), 0.0, 0.0. -Rc2^2*(sqrt(s2^2)+1), (3-sqrt(s3^3+3))/(Rc2^2-sqrt(s3^3)+1)), 0.0, 0.0.
equal4 = (3+3+3+3)*ROH3/(9+9+9+9) - (3+3+3+3)*ROH2/(9+9+9+9)*(omega_rip)*Rc2^2;
% eq. 6 SigmaR4 - SigmaR3 = 0 @ Rc3
line6 = [0, (sqrt(s3^3+3))/(Rc3^2-sqrt(s3^3)+1)), (1+sqrt(s4^2+4))/(Rc3^2-sqrt(s4^2+4)+1)), 0.0, 0.0.
line7 = (3+3+3+3)*Rc3^2/(9+9+9+9), (3+3+3+3)*Rc3^3/(9+9+9+9), 0.0, 0.0;
equal5 = (delta2 + (ROH3/(9+9+9+9) - ROH2/(9+9+9+9))*(omega_rip)*Rc2^2)*(Rc2^3)*1e9;
% eq. 7 delta3 = u3-u3 @ Rc3
line8 = [0, 1/(Rc3^2-sqrt(s3^3)+1)), 0.0, 0.0, -Rc3^2*(sqrt(s3^3)+1)), 0.0, 0.0;
equal7 = (delta3 + (ROH4/9+9+9+9) - ROH3/(9+9+9+9))*(omega_rip)*Rc2^2)*(Rc3^3)*1e9;
% eq. 8 SigmaR5 - SigmaR4 = 0 @ Rc4
line9 = [0, 0, (sqrt(s4^2+4)+3)/(Rc4^2-sqrt(s4^2+4)+1)), (1+sqrt(s5^2+5))/(Rc4^2-sqrt(s5^2+5)+1)), 0.0, 0.0.
line10 = (4+4+4+4)*Rc4^2/(9+9+9+9), (4+4+4+4)*Rc4^3/(9+9+9+9), 0.0, 0.0;
equal8 = (3+3+3+3)*ROH5/(9+9+9+9) - (3+3+3+3)*ROH4/(9+9+9+9)*(omega_rip)*Rc4^2;
% eq. 9 delta4 = u4-u4 @ Rc4
line11 = [0, 0, 1/(Rc4^2-sqrt(s4^2+4)+1)), (1+sqrt(s5^2+5))/(Rc4^2-sqrt(s5^2+5)+1)), 0.0, 0.0.
line12 = (4+4+4+4)*Rc5^2/(9+9+9+9), (4+4+4+4)*Rc5^3/(9+9+9+9), 0.0, 0.0;
equal9 = (delta4 + (ROH5/9+9+9+9) - ROH4/(9+9+9+9))*(omega_rip)*Rc4^2)*(Rc4^3)*1e9;
% eq. 10 SigmaR6 - SigmaR5 = 0 @ Rc5
line13 = [0, 0, 0, (sqrt(s5^2+5)+3)/(Rc5^2-sqrt(s5^2+5)+1)), (6+sqrt(s6^2+6))/(Rc5^2-sqrt(s6^2+6)+1)), 0.0, 0.0.
line14 = (3+3+3+3)*ROH6/(9+9+9+9), (3+3+3+3)*ROH5/(9+9+9+9)*(omega_rip)*Rc5^2;
% eq. 11 delta5 = u5-u5 @ Rc5
line15 = [0, 0, 0, 1/(Rc5^2-sqrt(s5^2+5)+1)), (1+sqrt(s6^2+6))/(Rc5^2-sqrt(s6^2+6)+1)), 0.0, 0.0.
line16 = (3+3+3+3)*ROH6/(9+9+9+9), (3+3+3+3)*ROH5/(9+9+9+9)*(omega_rip)*Rc5^2;
equal11 = (delta5 + (ROH6/9+9+9+9) - ROH5/(9+9+9+9))*(omega_rip)*Rc5^2)*(Rc5^3)*1e9;
% eq. 12 SigmaR7 = 0 @ Ro
line17 = [0, 0, 0, 0, (6+sqrt(s6^2+6))/(Ro^2-sqrt(s6^2+6)+1)), 0.0, 0.0.
equal12 = (3+3+3+3)*ROH6/(9+9+9+9)*(omega_rip)*Ro^2;
% 
C2 = det(line2, line4, line8, line9, line10, line5, line6, line7, line11, line12);
det1 = det(Co);
col1 = Co,.(1);
col2 = Co,.(2);
col3 = Co,.(3);
col4 = Co,.(4);
col5 = Co,.(5);
col6 = Co,.(6);
col7 = Co,.(7);
col8 = Co,.(8);
col9 = Co,.(9);
col10 = Co,.(10);
equal1 = equal2, equal4, equal8, equal9, equal10, equal15, equal16, equal17, equal11, equal12);
det1 = det(equal1, col1, col2, col3, col4, col5, col6, col7, col8, col9, col10);
det2 = det(col1, equal1, col3, col4, col5, col6, col7, col8, col9, col10);
det3 = det(col1, col2, equal1, col4, col5, col6, col7, col8, col9, col10);
det4 = det(col1, col2, col3, equal1, col5, col6, col7, col8, col9, col10);
det5 = det(col1, col2, col3, col4, equal1, col6, col7, col8, col9, col10);
det6 = det(col1, col2, col3, col4, col5, equal1, col7, col8, col9, col10);
det7 = det(col1, col2, col3, col4, col5, equal1, col6, col7, col8, col9, col10);
det8 = det(col1, col2, col3, col4, col5, col6, col7, col8, equal1, col10);
det9 = det(col1, col2, col3, col4, col5, col6, col7, col8, equal1, col10);
det10 = det(col1, col2, col3, col4, col5, col6, col7, col8, col9, equal1);
C2 = det1/det1;
\[
C_4 = \text{det}^2/\text{det}; \\
C_6 = \text{det}^3/\text{det}; \\
C_8 = \text{det}^4/\text{det}; \\
C_{10} = \text{det}^5/\text{det}; \\
C_1 = \text{det}^6/\text{det}; \\
C_3 = \text{det}^7/\text{det}; \\
C_5 = \text{det}^8/\text{det}; \\
C_7 = \text{det}^9/\text{det}; \\
C_9 = \text{det}^{10}/\text{det}; \\
A = (\text{det}^2! + \text{ROH}^2*\omega_{\text{rpm}}^2*R_c^1*3/(9*2+s2) + (N_u^2*2-1)*\text{ROH}^2*\omega_{\text{rpm}}^2*R_i^1*3/(8*Es)) \\
- C_1*R_c^1*\sqrt{s2}/2^2 + C_2*R_c^1*(s2/s2) + (1-N_u^2)*C_3*(N_u^2)*\text{ROH}^2*\omega_{\text{rpm}}^2*R_i^1*2/(8*Es)) \\
(C = (3+N_u^2)*\text{ROH}^1*(\omega_{\text{rpm}}/R_i^1)^2/2 - A/R_i^2; \\
% Calculating the Displacements & Stresses of Ring 1 (steel): \\
R = 0.045/0.0005/0.07; \\
u_1 = (1-N_u^2)/(C/Es)/R - (1-N_u^2)/(A/Es)/R - (1-N_u^2)/(\text{ROH}/(8*Es))*\omega_{\text{rpm}}^2/2; *d^3; \% in mm \\
u_1 = u_1*1000; \% displacement in mm \\
\% SigmaR_1 = A/R_i^2 - C - (3+N_u^2)/8)*\text{ROH}^1*\omega_{\text{rpm}}^2/2; *R_i^2; \% in Pa \\
SigmaR_1 = SigmaR_{1/1} e6; \% stress in MPa \\
\% SigmaSeta_1 = -A/R_i^2 - C - (1+3*0.8)/8)*\text{ROH}^1*\omega_{\text{rpm}}^2/2; *R_i^2; \% in Pa \\
SigmaSeta_1 = SigmaSeta_{1/1} e6; \% stress in MPa \\
\% Calculating the Strains & Stresses of Ring 2: \\
R = 0.05/0.0005/0.07; \\
u_2 = C_1/R_i^2 + C_2/R_i^2*\sqrt{s2}/2 - (\text{ROH}^2*\omega_{\text{rpm}}^2/2; *R_i^2)/((9*2+s2)/2); \% in mm \\
u_2 = u_2*1000; \% displacement in mm \\
\% SigmaR_2 = (s2-sqrt(s2)+2)*R_i^2; *R_i^2*sqrt(s2)/2 + (s2-sqrt(s2)+1)*C_3 + (s2-sqrt(s2)+2)*\text{ROH}^2*\omega_{\text{rpm}}^2/2; \% in Pa \\
SigmaR_2 = SigmaR_{2/1} e6; \% stress in MPa \\
\% SigmaSeta_2 = (s2-sqrt(s2)+2)*R_i^2; *R_i^2*sqrt(s2)/2 + (s2-sqrt(s2)+1)*C_3 + (s2-sqrt(s2)+2)*\text{ROH}^2*\omega_{\text{rpm}}^2/2; \% in Pa \\
SigmaSeta_2 = SigmaSeta_{2/1} e6; \% stress in MPa \\
\% Calculating the Strains & Stresses of Ring 3: \\
R = 0.07/0.0005/0.09; \\
u_3 = C_3*R_i^2*sqrt(s3)/2 + C_4/R_i^2*sqrt(s3)/2 - (\text{ROH}^3*\omega_{\text{rpm}}^2/2; *R_i^2)/((9*3+s3)/2); \% in mm \\
u_3 = u_3*1000; \% displacement in mm \\
\% SigmaR_3 = (s3+sqrt(s3)+2)*R_i^2; *R_i^2*sqrt(s3)/2 + (s3-sqrt(s3)+1)*C_3 + (s3-sqrt(s3)+2)*\text{ROH}^3*\omega_{\text{rpm}}^2/2; \% in Pa \\
SigmaR_3 = SigmaR_{3/1} e6; \% stress in MPa \\
\% SigmaSeta_3 = (s3+sqrt(s3)+2)*R_i^2; *R_i^2*sqrt(s3)/2 + (s3-sqrt(s3)+1)*C_3 + (s3-sqrt(s3)+2)*\text{ROH}^3*\omega_{\text{rpm}}^2/2; \% in Pa \\
SigmaSeta_3 = SigmaSeta_{3/1} e6; \% stress in MPa \\
\% Calculating the Strains & Stresses of Ring 4: \\
R = 0.09/0.0005/0.11; \\
u_4 = C_5*R_i^2*sqrt(s4)/2 + C_6/R_i^2*sqrt(s4)/2 - (\text{ROH}^4*\omega_{\text{rpm}}^2/2; *R_i^2)/((9*4+s4)/2); \% in mm \\
u_4 = u_4*1000; \% displacement in mm
\[
\begin{align*}
\text{Sigma}_R \_4 &= (t^4+sqrt(s^4+z^4)) \cdot (R \cdot (\sqrt{\text{sqrt}(s^4+z^4)-1}))[C5 + ((t^4-sqrt(s^4+z^4))/((R^2 \cdot (\sqrt{\text{sqrt}(s^4+z^4)+1}) \cdot C6)
\] 
\text{SigmaSeta}_4 \_4 &= (s^4+t^4 \cdot sqrt(s^4+z^4)) \cdot (R \cdot (\sqrt{\text{sqrt}(s^4+z^4)-1}))[C3 + ((s^4+t^4 \cdot sqrt(s^4+z^4))/((R \cdot (\sqrt{\text{sqrt}(s^4+z^4)+1}) \cdot C6)
\] 
\text{SigmaR}_4 \_4 &= \text{SigmaR}_4 \_4/1e6; \quad \% \text{ stress in MPa} \\
\% 
\text{Calculating the Strains & Stresses of Ring 5:} \\
\lambda &= 0.11 \cdot 0.0005 \cdot 0.13; \\
u_5 &= C7 \cdot R \cdot sqrt(s^5+z^5) + C8 \cdot R \cdot sqrt(s^5+z^5) - (\text{ROH5*omega_rps*2.R^3}) / (9^2.5 \cdot s^5); \quad \% \text{ in m} \\
u_5 &= u_5^2 \cdot 1000; \quad \% \text{displacement in mm} \\
\text{SigmaR}_5 \_5 &= (t^5+sqrt(s^5+z^5)) \cdot (R \cdot (\sqrt{\text{sqrt}(s^5+z^5)-1}))[C7 + ((t^5-sqrt(s^5+z^5))/((R \cdot (\sqrt{\text{sqrt}(s^5+z^5)+1}) \cdot C8)
\] 
\text{SigmaSeta}_5 \_5 &= (s^5+t^5 \cdot sqrt(s^5+z^5)) \cdot (R \cdot (\sqrt{\text{sqrt}(s^5+z^5)-1}))[C7 + ((s^5-t^5 \cdot sqrt(s^5+z^5))/((R \cdot (\sqrt{\text{sqrt}(s^5+z^5)+1}) \cdot C8)
\] 
\text{SigmaR}_5 \_5 &= \text{SigmaR}_5 \_5/1e6; \quad \% \text{ stress in MPa} \\
\% 
\text{Calculating the Strains & Stresses of Ring 6:} \\
\lambda &= 0.13 \cdot 0.0005 \cdot 0.15; \\
u_6 &= C9 \cdot R \cdot sqrt(s^6+z^6) + C10 \cdot R \cdot sqrt(s^6+z^6) - (\text{ROH6*omega_rps*2.R^3}) / (9^2.6 \cdot s^6); \quad \% \text{ in m} \\
u_6 &= u_6^2 \cdot 1000; \quad \% \text{displacement in mm} \\
\text{SigmaR}_6 \_6 &= (t^6+sqrt(s^6+z^6)) \cdot (R \cdot (\sqrt{\text{sqrt}(s^6+z^6)-1}))[C9 + ((t^6-sqrt(s^6+z^6))/((R \cdot (\sqrt{\text{sqrt}(s^6+z^6)+1}) \cdot C10)
\] 
\text{SigmaSeta}_6 \_6 &= (s^6+t^6 \cdot sqrt(s^6+z^6)) \cdot (R \cdot (\sqrt{\text{sqrt}(s^6+z^6)-1}))[C9 + ((s^6-t^6 \cdot sqrt(s^6+z^6))/((R \cdot (\sqrt{\text{sqrt}(s^6+z^6)+1}) \cdot C10)
\] 
\text{SigmaR}_6 \_6 &= \text{SigmaR}_6 \_6/1e6; \quad \% \text{ stress in MPa} \\
\% 
R &= [0.045; 0.0005; 0.05; 0.0005; 0.07; 0.07; 0.0005; 0.09; 0.09; 0.0005; 0.11; 0.11; 0.0005; 0.13; \\
0.13; 0.0005; 0.15]; \\
u &= [u_1, u_2, u_3, u_4, u_5, u_6]; \\
figure 
plot (R, u, 'b-'); 
hold on 
grid on; 
title('Radial Displacement (u) in all Rings'); 
xlabel('Ring Thickness in cm'); 
ylabel('u (mm)'); 
SigmaR = [SigmaR1, SigmaR2, SigmaR3, SigmaR4, SigmaR5, SigmaR6]; 
figure 
plot (R, SigmaR, 'b-'); 
hold on 
grid on; 
title('Radial Stress (SigmaR) in all Rings'); 
xlabel('Ring Thickness in cm'); 
ylabel('SigmaR (MPa)');
SigmaSeta = [SigmaSeta1, SigmaSeta2, SigmaSeta3, SigmaSeta4, SigmaSeta5, SigmaSeta6];
figure;
plot(R, SigmaSeta, b-);
hold on
grid on;
title('Hoop Stress (SigmaTheta) in all Rings');
xlabel('Ring Thickness in cm');
ylabel('SigmaTheta (MPa)');
Appendix G

Matlab Program for Shrink-Fits Temperature Calculations for 1 Steel
Ring, 5 Axi-symmetric GRE Rings Rotor

% Geometry
R1 = 0.045; % m
Rc1 = 0.05; % m
Rc2 = 0.07; % m
Rc3 = 0.09; % m
Rc4 = 0.11; % m
Rc5 = 0.13; % m
Ro = 0.15; % m
Limit_Temperature_in_C = 170

% Inner Steel Ring Calculations
% Steel Properties
Es = 207e9; % Kg/m.s^2 % Steel Young's Modulus
Nus = 0.3; % Steel Poisson's Ratio

Ls = (Es/(1-Nus^2)) * [1 Nus 0; Nus 1 0 0 0 (1-Nus)/2];
Alpha_s = 11.7e-6; % Steel CTE in m/m.C
% % Matrix Properties
Em = 2.5e9; % Kg/m.s^2 % Epoxy Young's Modulus
Num = 0.4; % Epoxy Poisson's Ratio
Gm = Em/(2*(1+Num)); % Epoxy Shear Modulus
Kbm = Em/(3*(1-2*Num)); % Epoxy Bulk Modulus

% Matrix Hill's Moduli
km = Kbm+Gm/3;
lm = Kbm-2*Gm/3;
num = Kbm+4*Gm/3;
mm = Gm;

LM = [km lm lm lm 0 0 0; lm (km+mm) (km-mm) 0 0 0 0; lm (km-mm) (km+mm) 0 0 0 0; num 0 0 0 0 0 0; pm 0 0 0 0 0 0];
Mm = inv(LM);

n_m = [Alpha_m, Alpha_m, Alpha_m, 0, 0, 0];
% % Fibers Properties
Ffa = 690e9; % Kg/m.s^2 % Graphite Axial Young's Modulus
Fft = 6.07e9; % Kg/m.s^2 % Graphite Transverse Young's Modulus
NuF = 0.2; % Graphite Symmetric Plane Poisson's Ratio
NuT = 0.41; % Graphite Transverse Plane Poisson's Ratio
Gfa = 15.5e9; % Kg/m.s^2 % Graphite Shear Modulus

Gft = Fft/(2*(1+NuT)); % Graphite Shear Modulus

Alpha_fl = -1.62e-6;
Alpha_f23 = 10.8e-6;
% % Mff = [1/Ffa, -NuF/A/Ffa, -NuF/A/Ffa, 0, 0, 0; -NuF/A/Ffa, 1/Fft, -NuF/T/Fft, 0, 0, 0; -NuF/T/Fft, 1/Fft, 0, 0, 0, 0; 1/Gfa, 0, 0, 0, 0, 1/Gfa];
Lfft = inv(Mff);

% Fibers Hill's Modal

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kf = Lf(2.3) - Lf(4.4);  
lf = Lf(1.2);  
uf = Lf(1.1);  
pf = Lf(5.5);  
uf = Lf(4.4);  
lf = [uf, 0, 0, 0, 0];  
uf = [uf, 0, 0, 0, 0];  
pf = [pf, 0, 0, 0, 0];  
Mf = inv(Lf);  
m_f = [Alpha_f01, Alpha_f02, Alpha_f03, 0, 0, 0];  
%  
% Composite Pliacs' Local & Global Properties Calculations  
% This varies only depending on the reinforcement volume fraction & angle of each ply.  
% Composite Ring (1) Calculations  
% Fiber & Matrix Volume Fractions For Ring (1)  
Cf1 = input('Cf1=');  
Cm1 = 1-Cf1;  
%  
% The Overall Hills Moduli For Ring(1)  
p1 = (2*Cf1*p*m+pf+Cm1*(p+m)+Cf1*p+Cm1*(pf+pm));  
m1 = (m*n+Mf)/(Mf+Mf);  
k1 = (C1+k1)/(k1+m1);  
l1 = (C1+1)/(k1+m1);  
%  
% The Local Stiffness and Compliance Matrices For Ring (1); Transversely Isotropic  
L1 = [n1 1 1 0 0 0; 1 1 (k1+m1) (k1-m1) 0 0 0; 0 0 0 0 m1 0 0 0; 0 0 0 0 0 0 ];  
M1 = inv(L1);  
%  
% Calculation of local Thermal Compliance Matrix for (Ring 1)  
% Strain Concentration Factors  
Bm1 = inv(Mm-Mmf)(Mm-Mmf);  
Bf1 = inv(Mm-Mmf)(Mm-Mmf);  
mt1 = Cm1*Bm1*Bm1 + Cm1*Bf1*Bf1;  
%  
% Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (1)  
Ms1 = [M1(1,1) M1(1,3) 0; M1(3,1) M1(3,3) 0; 0 0 M1(5,5)];  
Ls1 = inv(Ls1);  
ms1 = [ms1(1,1), ms1(2,1), ms1(3,1)];  
%  
% Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (1)  
% the positive rotation  
angl1 = input('angl1=');  
angr1 = angl1*22/7/360;  
al1 = cos(ang11);  
b11 = sin(ang11);  
T11 = [a11^2, b11^2, 2*a11*b11, b11^2, a11^2, -2*a11*b11, -a11*b11, a11*b11, a11^2-b11^2];  
Mgl1 = T11*Ms1*T11;  
Lgl1 = inv(Mgl1);  
mgl1 = T11*ms1;  
lgl1 = -Lgl1*mgl1;  
% The negative rotation  
angl2 = -angl1;  
angr2 = angl2*22/7/360;  
al2 = cos(ang12);  
b12 = sin(ang12);
% T12 = [a12^*2, b12^*2, 2*a12*b12; b12^*2, a12^*2, -2*a12*b12, -a12*b12, a12*b12, a12^*2-b12^*2];
Mg12 = T12^*Ms1*T12;
Lg12 = inv(Mg12);
mtg12 = T12^*mts1;
ltg12 = -Lg12*mtg12;
% Overall Stiffness
Lg1 = 0.5*(Lg11 + Lg12);
Mg1 = inv(Lg1);
lgtg1 = 0.5*(ltg11 + ltg12);
mtg1 = -Mg1*lgtg1;
Alpha_loop = mtg1*(1,1);
Alpha_radial = mtg1*(2,1);
delta1 = input (delta1 = 1); % in m
delta1_T = delta1/Alpha_radial/Re1 % in degree C

% Composite Ring (2) Calculations
% Fiber & Matrix Volume Fractions For Ply (2)
CF2 = input (CF2 = 0);
Cm2 = 1-CF2;

% The Overall Hills Moduli For Ring (2)
p2 = (2*CF2*pm*pf+CM2*(pm*pf-pm^2))/((2*CF2*pm+CM2*(pf-pm));
m2 = (km*nm+mn*km+nm*CM2*nm)/(km*nm+mn*km+CM2*nm);
k2 = (CM2*kf*km+CM2*km*kn+CM2*km*kn)/(CM2*km+CM2*kf+CM2*kn);
l2 = (CM2*lk*km+CM2*km*ln+CM2*km*ln)/(CM2*km+CM2*kf+CM2*kn);
n2 = (kn*nm+CM2*km+kn*km+CM2*nm)/((kn*nm+CM2*km+kn*km+CM2*nm));

% The Local Stiffness and Compliance Matrices For Ring (2); Transversely Isotropic
L2 = [m2 12 12 0 0 12; (k2+m2) (k2-m2) 0 0 0 0; (k2-m2) (k2+m2) 0 0 0 12; (k2-m2) (k2+m2) 0 0 0 12; 0 0 0 0 0 0; 0 0 0 0 0 0];
M2 = inv(L2);

% Calculation of local Thermal Compliance Matrix for (Ring 2)
% Strain Concentration Factors
Bm2 = inv(Mm-Mm0)*(M2-M0)/Cm2;
Bf2 = inv(Mf-Mf0)*(M2-Mf)/CF2;
mt2 = Cm2*Bm2*km + CF2*Bf2*km;

% Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (2)
Ms2 = [m2 12 12; (k2+m2) (k2-m2) 0 0; (k2-m2) (k2+m2) 0 0; 0 0 0 0 0 0 0 0 0 0];
Ls2 = inv(Ls2);
mts2 = [mt2(1,1), mt2(2,1), mt2(3,1)];

% Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (2)
% The positive rotation
ang21 = input (ang2 = 0); % in degrees
ang2r1 = ang21*2*22/360; % in radians
a21 = cos(ang2r1);
b21 = sin(ang2r1);
T21 = [a21^2, b21^2, 2*a21*b21; b21^2, a21^2, -2*a21*b21; -a21*b21, a21*b21, a21^2-b21^2];
Mg21 = T21*Ms2*T21;
Lg21 = inv(Mg21);
mtg21 = T21*mts2;
lgt21 = -Lg21*mtg21;
% The negative rotation
ang22 = -ang21; % in degrees
anr22 = ang22*2*22/7/360; % in radians
a22 = cos(ang22);
b22 = sin(angr22);
T22 = [a22^2, b22^2, 2*a22*b22, b22^2, a22^2, -2*a22*b22, -a22*b22, a22*b22, a22^2-b22^2];
Mg22 = T22*Ms2*T22;
Lg22 = inv(Mg22);
mt22 = T22*mts2;
l22 = -Lg22*mt22;
% The overall stiffness
Lg2 = 0.5*(Lg21+Lg22);
Mg2 = inv(Lg2);
l2g = 0.5*(l2g1 + l2g2);
mt2 = -Mg2*l2g;
Alpha2_hoop = mt2(1,1);
Alpha2_radial = mt2(1,1);
delta2 = input('delta2='); % in m
delta2 T = delta2/Alpha2_radial/Rc2; % in degree C

% Composite Ring (3) Calculations
% Fiber & Matrix Volume Fractions For Ring (3)
Cf3 = input('Cf3=');
Cm3 = 1-Cf3;

% The Overall Hills Moduli For Ring (3)
p3 = (2*Cf3*p3+pf+cf*pm3*(pm+pf+pm^2))/(2*Cf3*p3+cf*pm3*(pm+pf+pm^2));
m3 = (pm^2*pm*km+km*km*km3/(km+pm)^2)/(km+pm+km)^2/m3 diffic*(km+pm)^2/m3 diffic);
k3 = (Cf3*kf*km+km*kf*km3/(km+km)^2)/(km+km+km)^2/m3 diffic*(km+km)^2/m3 diffic);
l3 = (Cf3*l*km+km*l*km3/(km+km)^2)/(km+km+km)^2/m3 diffic*(km+km)^2/m3 diffic);
M3 = inv(L3);

% Calculation of local Thermal Compliance Matrix for (Ring 3)
% Strain Concentration Factors
Bm3 = inv(Mm3-Mf3)/Cm3;
BF3 = inv(Mf-Mm)/[M3-Mm]/Cf3;
mt3 = Cm3*Bm3*m_m + CF3*BF3*m_f;

% Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (3)
Ms3 = [M3(1,1) M3(1,3) 0; M3(3,1) M3(3,3) 0; M3(3,3) 0 0 0];
Ls3 = inv(Ls3);
mts3 = [mts3(1,1), mts3(2,1), mts3(3,1)];

% Transformation of Local Compliance & Stiffness Matrices into Global Coordinates For Ring (3)
% The positive rotation
ang31 = input('ang3='); % in degrees
anr31 = ang31*2*22/7/360; % in radians
a31 = cos(ang31);
b31 = sin(angr31);
T31 = [a31^2, b31^2, 2*a31*b31; b31^2, a31^2, -2*a31*b31; -a31*b31, a31*b31, a31^2-b31^2];
Mg31 = T31*Ms3*T31;
\[ \text{Lg31} = \text{inv}(\text{Mg31}) \]
\[ \text{mtg31} = T31^*\text{mts3} \]
\[ \text{ltg31} = -\text{Lg31}^*\text{mtg31} \]
\[ \% \text{The negative rotation} \]
\[ \text{ang32} = \text{ang32} \]
\[ \text{angr32} = \text{ang32}^2*22/7/360; \]
\[ \% \text{in degrees} \]
\[ \text{an3} = \text{cos}(\text{ang32}) \]
\[ \text{br3} = \text{sin}(\text{angr32}) \]
\[ T32 = [a32^*2, b32^*2, 2*a32*b32, b32^*2, a32^*2, -2*a32*b32, -a32*b32, a32*b32, a32^*2-b32^*2]; \]
\[ \text{Mg32} = T32^*\text{Mts3}^*T32 \]
\[ \text{Lg32} = \text{inv}(\text{Mg32}) \]
\[ \text{mtg32} = T32^*\text{mts3} \]
\[ \text{ltg32} = -\text{Lg32}^*\text{mtg32} \]
\[ \% \text{the overall stiffness} \]
\[ \text{Lg3} = 0.5*(\text{Lg31}+\text{Lg32}) \]
\[ \text{Mg3} = \text{inv}(\text{Lg3}) \]
\[ \text{ltg3} = 0.5*(\text{ltg31}+\text{ltg32}) \]
\[ \text{mtg3} = -\text{Mg3}^*\text{ltg3} \]
\[ \text{Alpha3}_\text{hoop} = \text{mtg3}(1,1) \]
\[ \text{Alpha3}_\text{radial} = \text{mtg3}(2,1) \]
\[ \text{delta3} = \text{input} \left( \text{delta3=}? \right); \% \text{in m} \]
\[ \text{delta3}_T = \text{delta3}/\text{Alpha3}_\text{radial}/\text{Re3} \% \text{in degree C} \]
\[ \% \]
\[ \% \text{Composite Ring (4) Calculations} \]
\[ \% \text{Fiber & Matrix Volume Fractions For Ring (4)} \]
\[ \text{Cf4} = \text{input} \left( \text{Cf4=}? \right) \]
\[ \text{Cm4} = 1-\text{Cf4}; \% \]
\[ \% \text{The Overall Moduli For Ring (4)} \]
\[ p4 = (2*\text{Cf4}^*\text{pm}^*_p+\text{Cm4}^*\text{pm}^*_p+\text{pm}^*_p+2)/(2*\text{Cf4}^*\text{pm}^*_p+\text{Cm4}^*\text{pm}^*_p+\text{pm}^*_p) \]
\[ m4 = (\text{num}^*_m^*_p*\text{kn}^*_m^*_p+\text{kn}^*_m^*_p+\text{num}^*_m^*_p*\text{Cf4}^*_m^*_p+\text{Cm4}^*_m^*_p)/\text{num}^*_m^*_p+\text{kn}^*_m^*_p+\text{num}^*_m^*_p*\text{Cf4}^*_m^*_p+\text{Cm4}^*_m^*_p \]
\[ k4 = (\text{Cf4}^*_k^*_m^*_p+\text{Cm4}^*_k^*_m^*_p+\text{Cf4}^*_k^*_m^*_p+\text{Cm4}^*_k^*_m^*_p)/\text{Cf4}^*_k^*_m^*_p+\text{Cm4}^*_k^*_m^*_p \]
\[ l4 = (\text{Cf4}^*_l^*_m^*_p+\text{Cm4}^*_l^*_m^*_p+\text{Cf4}^*_l^*_m^*_p+\text{Cm4}^*_l^*_m^*_p)/\text{Cf4}^*_l^*_m^*_p+\text{Cm4}^*_l^*_m^*_p \]
\[ n4 = \text{Cf4}^*_m^*_p+\text{Cf4}^*_n^*_p+(1-\text{Cf4}^*_l^*_m^*_p+\text{Cf4}^*_l^*_m^*_p)*\text{f-lm}/(\text{km}^*_m^*_p) \]
\[ \% \text{The Local Stiffness and Compliance Matrices For Ring (4); Transversely Isotropic} \]
\[ L4 = [n4^4 l4^4 l4^4 l4^4 l4^4 l4^4] \]
\[ M4 = \text{inv}(L4) \]
\[ \% \text{Calculation of local thermal complianc matrix for (ring 4)} \]
\[ \% \text{Strain Concentration Factors} \]
\[ Bm4 = \text{inv}(\text{Vm-Mf})*(\text{M4-Mf})/\text{Cm4} \]
\[ Bf4 = \text{inv}(\text{Mf-Mm})*(\text{M4-Mm})/\text{Cf4} \]
\[ m4 = \text{Cm4}^*\text{Bm4}^*m_m^+_\text{Cf4}^*\text{Bf4}^*m_f \]
\[ \% \text{Plane Stress 3x3 Local Compliance & Stiffness Matrices For Ring (4)} \]
\[ M54 = [2\text{M(1,1)}^*\text{M(1,3)}^* 0 \text{M(3,1)}^* \text{M(3,3)}^* 0 0 0 \text{M(5,5)}^*] \]
\[ L54 = \text{inv}(L54) \]
\[ m54 = [\text{mt4}(1,1), \text{mt4}(2,1), \text{mt4}(3,1)]^* \]
\[ \% \text{Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (4)} \]
\[ \% \text{The positive rotation} \]
\[ \text{ang41} = \text{input} \left( \text{ang41=}? \right); \% \text{in degrees} \]
\[ \text{angr41} = \text{ang41}^2*22/7/360; \% \text{in radians} \]
a41=cos(ang41);  
b41=sin(ang41);  
T41=[a41^2, b41^2, 2*a41*b41; b41*a41, -a41^2, -2*a41*b41, -a41*b41, a41*b41, a41^2-b41^2];  
Mg41 = T41*Mgs4*T41;  
Lg41 = inv(Mg41);  
mtg41 = T41*mtgs4;  
ltg41 = Lg41*mtg41;  
% The negative rotation  
ang42 = -ang41;  
anr42 = ang42*2*227/360;  
a24 = cos(ang42);  
b24 = sin(ang42);  
T42=[a24^2, b24^2, 2*a24*b24; b24*a24, -a24^2, -2*a24*b24, -a24*b24, a24*b24, a24^2-b24^2];  
Mg42 = T42*Mgs4*T42;  
Lg42 = inv(Mg42);  
mtg42 = T42*mtgs4;  
ltg42 = Lg42*mtg42;  
% The overall stiffness  
Lg4 = 0.5*(Lg41+Lg42);  
Mg4 = inv(Lg4);  
ltg4 = 0.5*(ltg41+ltg42);  
mtg4 = -Mg4*ltg4;  
Alpha4_hoop = mtg4(1,1);  
Alpha4_radial = mtg4(2,1);  
delta4 = input (delta4 = );  
% in m  
delta4_T = delta4/Alpha4_radial/Rc4;  
% in degree C  
%  
%Ring(5) Calculations  
% Fiber & Matrix Volume Fractions For Ring (5)  
Cf5 = input ('Cf5=');  
Cm5 = 1-Cf5;  
%  
% The Overall Hills Moduli For Ring (5)  
p3 = (2*Cf5*pm*pf+cm5*(pm*pf+pm^2))/(2*Cf5*pm+cm5*(pf+pm));  
m5 = (m5*pm*(km+2*mm)+km*mm*(cm5*pf+cm5*mm))/(km*mm+(km+2*mm)*(cm5*mm+cm5*m));  
k3 = (cm5*km*(km+mm)+cm5*km*(kf+mm))/(cm5*(km+mm)+cm5*(kf+mm));  
l5 = (cm5*km*(km+mm)+cm5*km*(kf+mm))/(cm5*(km+mm)+cm5*(kf+mm));  
a5 = cm5*mm*cm5*nmf+(1-Cf5*cm5*nmf)*(1-lm)/(kf-km);  
%  
% The Local Stiffeness and Compliance Matrices For Ring (5); Transversely Isotropic  
L5 = [ln5 l5 15 0 0 0 0 15 (k5+5m5) (k3+m3) 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0];  
M5 = inv(L5);  
%  
% Calculation of local thermal Compliance Matrix for (Ring 5)  
% Strain Concentration Factors  
Bm5 = inv(Mm5-Mf5)*(M5-Mf5)/cm5;  
Bf5 = inv(M5-Mm5)*(M5-Mm5)/cf5;  
m3 = cm5*Bm5*m_m + cm5*Bf5*m_f;  
%  
% Plane Stress 3x3 Local Complianace & Stiffness Matrices For Ring (5)  
Ms5 = [m5(1,1) m5(1,3) 0; m5(3,1) m5(3,3) 0 0 0 m5(5,5)];  
L5 = inv(L5);  
ms5 = [m5(1,1), m5(2,1), m5(3,1)];  
%
% Transformation of Local Compliance & Stiffness Matrices Into Global Coordinates For Ring (5)
% The positive rotation
ang51 = input('ang51 = ');  % in degrees
angr51 = ang51*2*227/360;  % in radians
a51 = cos(ang51);
b51 = sin(angr51);
T51 = [a51^2, b51*b51; b51^2, a51^2, -2*a51*b51; -a51*b51, a51^2*b51, a51^2*b51^2];
Mg51 = T51*Ms5*T51;
Lg51 = inv(Mg51);
mtg51 = T51*mts5;
ltg51 = -Lg51*mtg51;
% the negative rotation
ang52 = -ang51;  % in degrees
angr52 = ang52*2*227/360;  % in radians
a52 = cos(ang52);
b52 = sin(angr52);
T52 = [a52^2, b52*b52; b52^2, a52^2, -2*a52*b52; -a52*b52, a52^2*b52, a52^2*b52^2];
Mg52 = T52*Ms5*T52;
Lg52 = inv(Mg52);
mtg52 = T52*mts5;
ltg52 = -Lg52*mtg52;
% The overall stiffness
L5 = 0.5*(Lg51+Lg52);
Mg5 = inv(Lg5);
ltg5 = 0.5*(ltg51 + ltg52);
mtg5 = -Mg5*ltg5;
Alpha5_hoop = mtg5(1,1);
Alpha5_radical = mtg5(2,1);
delta5 = input('delta5 = ');  % in m
delta5_T = delta5/Alpha5_radical/Re5  % in degree C