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Dyonic Taub-NUT-AdS Space Phase Structure

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Dyonic Taub-NUT-AdS Space Phase Structure

A THESIS SUBMITTED BY

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TO THE

Physics Graduate Program

SUPERVISED BY

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May 14, 2023

in partial fulfillment of the requirements for the degree of Master of Science in Physics

Declaration of Authorship

I, Mohamed Tharwat declare that this thesis titled, “Dyonic Taub-NUT-AdS Spaces Phase Structure” and the work presented in it are my own. I confirm that:

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Abstract

The Taub-NUT spacetime remains to hold many mysteries more than half a centur/y after its discovery. The metric's controversy owes largely to the nut charge and the existence of Misner strings. Traditionally the metric is treated in the euclidean signature, this treatment hides the Misner strings. We treat the Taub-NUT spacetime with the Misner strings visible, not enforcing the time periodicity condition. We examine the phase structure belonging to three different horizon geometries. We deal with the hyperbolic, flat and spherical cases. We consider the stable phases, the phase transitions that exist between them, and find the preferable phases in all three spacetimes.

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∇	Covariant Derivative
d	Exterior Derivative
$\wedge$	Wedge Product
$R_{\mu\nu}$	Ricci Tensor
R	Ricci Scalar
$\epsilon_{\mu\nu\alpha\beta}$	Levi-Civita Tensor
$\Gamma^{\alpha}_{\mu\nu}$	Christoffel's Symbol of The Second Kind
$\mathcal{L}$	Lagrangian
c_p	Isobaric Heat Capacity
κ	Isothermal Compressibility

Chapter 1

General Relativity - an Introduction

During this chapter we will be introducing general relativity quite briefly, with a focus on the Einstein field equations. In the present and following chapter we will largely follow a mixture between the approaches presented in [1–3]. For readers interested in more detailed discussions, they should prove to be valuable resources.

1.1 Minkowski Spacetime

We will first discuss, briefly, the Minkowski spacetime before moving on to the Einstein field equations.

The shift from Newtonian mechanics to special relativity is a tale of trading invariances. Whereas Newtonian mechanics is invariant under Galilean transformations, special relativity is invariant under Lorentz transformations. We will quickly examine the difference in boosts within both.

Galilean invariance is inherently intuitive. Let there be relative linear motion between two objects. There exists a coordinate system in which said motion is purely in the x direction. Then we would have $y = y'$ and $z = z'$.

One would think that with time up on a pedestal, there would be no entertaining of a relationship between t and t' ; time is absolute. However, $t = t'$ if an only if both frames have coinciding origins at $t = t' = 0$ This is only a shift, though and does nothing to affect the absoluteness of time. This would result in the following transformation laws where u is the relative velocity between both frames and s is a constant.

$$\begin{aligned} x^{'} &= x - ut \\ t^{'} &= t + s \end{aligned} \tag{1.1}$$

As stated above, the transformations would be quite different for special relativity. The relationship in the two coordinates orthogonal to the direction of motion would be the same, but the x and t coordinates would transform differently.

$$\begin{aligned} x^{'} &= \gamma(x - \beta ct) \\ ct^{'} &= \gamma(ct - \beta x) \end{aligned} \tag{1.2}$$

where

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{1.3}$$

Let us first consider what the Minkowski metric represents. Speaking loosely, when we discuss a metric space is a space on which a distance function can be assigned between any two points. A differentiable manifold is a metric space that is everywhere differentiable and continuous. Now, lets us consider the Minkowski metric

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 \tag{1.4}$$

Under the assumption that the speed of light is invariant across inertial reference frames, it is straightforward to prove that distance on the Minkowski metric is invariant under Lorentz boosts. We will see later how other physical metrics revert to the Minkowski metric in the limit of weak gravitational fields.

1.2 Einstein Field Equations

We consider energy momentum tensor of a perfect fluid in Minkowski spacetime. Let the 4-velocity of the fluid be U^{μ}, ρ be the rest-frame density, and p the associated pressure. We can then write the tensorial equation for the energy-momentum tensor as

$$T_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} + p\eta_{\mu\nu} \tag{1.5}$$

For a perfect fluid in an co-moving reference frame, the off diagonal elements of the energy-momentum tensor $T^{\mu\nu}$ would be zero.[1] Since this equation is written in tensorial form, it should hold regardless of the metric in question. One way to produce general relativity is to replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$. This holds due to the minimal-coupling principle.[2]

We expect the energy-momentum of a perfect fluid to be a conserved quantity. In flat coordinates, this means that the divergence vanishes. To write it in a covariant form, we change the partial derivative into a covariant one.

$$\partial^{\mu} T_{\mu\nu} = 0 \longrightarrow \nabla^{\mu} T_{\mu\nu} = 0 \tag{1.6}$$

Having discussed a form for the energy-momentum tensor, we will opt for a Lagrangian derivation of the field equations. The action for a field takes the following form

$$S = \int d^n x \mathcal{L}(\Phi^i, \nabla_{\mu}\Phi^i) \tag{1.7}$$

Since we are operating in (possibly) curved space, the Lagrangian becomes a function of the fields and their covariant derivatives. It's also important to note that both $d^n x$ and $\mathcal{L}$ are both densities. While $-g$ is the determinant of the metric tesnor. This needs to happen if we want their product to be a tensor. To write the Lagrangian density as a true scalar we use the following

$$\mathcal{L} = \sqrt{-g}\hat{\mathcal{L}} \tag{1.8}$$

In the above equation, $\hat{\mathcal{L}}$ is truly a scalar. This allows us to express the action in terms of a scalar field by writing

$$S = \int d^n x \sqrt{-g}\hat{\mathcal{L}}(\Phi^i, \nabla_{\mu}\Phi^i) \tag{1.9}$$

We will consider the Einstein-Hilbert(EH) action. This action simply makes use of the only independent scalar that can be constructed from the metric where no derivatives of the metric that are higher than second order are involved. The EH action is then

$$S_{EH} = \int d^n x \sqrt{-g}\, R \tag{1.10}$$

We will now consider how this action varies under small variations of the metric tensor. It is useful to write the Ricci scalar as a contraction of the Ricci tensor and the inverse metric. The following relation between their variations becomes very useful.

$$
\begin{aligned}
g^{\mu\nu}g_{\mu\nu} &= \delta^{\mu}_{\nu} \\
\delta g_{\mu\nu} &= -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}
\end{aligned}
\tag{1.11}
$$

This shows us the a stationary point with respect to the metric will also be a stationary point with respect to its inverse. We can now make use of the inverse metric in calculating the action variation, which is mathematically less involved. Then action variation the becomes

$$
\begin{aligned}
\delta S_{EH} &= \int d^n x \delta(\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}\,) \\
&= \delta S_1 + \delta S_2 + \delta S_3
\end{aligned}
\tag{1.12}
$$

Where

$$
\begin{aligned}
\delta S_1 &= \int d^n x \delta \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \\
\delta S_2 &= \int d^n x \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} \\
\delta S_3 &= \int d^n x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}
\end{aligned}
\tag{1.13}
$$

The last part of the action will partially vanish on the assumption that the variation in the metric vanishes at infinity. However, the variation in the first derivative will not be zero, this will be more useful later but is ignorable for now. Thus we only consider δS_1 and δS_2 to make up the total of our variation.

$$
\delta S_{EH} = \int d^n x \sqrt{-g}(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\,)\delta g^{\mu\nu}\,)
\tag{1.14}
$$

Thus, the condition that the variation in the action vanishes for small variations in $\delta g^{\mu\nu}$ is

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0
\tag{1.15}
$$

This is none other than the Einstein equation in Vacuum.

We will now discuss what adding matter will do to the action and how we can, through that, define a stress-energy tensor. Taking S_m to be the action term contributed by matter we get

$$
S = \frac{1}{16\pi} S_{EH} + S_m
\tag{1.16}
$$

If we follow the same methodology, we would reach

$$\frac{1}{16\pi}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = T_{\mu\nu} \tag{1.17}$$

We can then define the energy momentum tensor to be

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{1.18}$$

As emphasised in [2], this results in a symmetric (0,2) tensor with the dimensions of density that satisfies

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{1.19}$$

Stressing that vacuum solutions are different from Minkowski spacetime is very important.

One important topic to mention is that of geodesics. We will focus on timelike geodesics. Timelike geodesics are the paths that a test particle undergoing no acceleration would follow. An affinely parameterised geodesic takes the following form

$$\frac{d^2x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0 \tag{1.20}$$

Geodesics make us see how the path for an object differs under the effect of gravity. If we were to be interested in the shape of orbits around the sun, we would be solving the field equations in vacuum. We would then take the resulting metric and examine geodesics that would correspond to orbits. The same is true for a solution of a spacetime outside a black hole.

Chapter 2

Black Hole Solutions

2.1 Stationary Solutions

Let us consider a solution to the field equations which would describe a spacetime where the observer would be sure that the gravitational or electromagnetic, fields will not be varying with time; this solution is stationary.

In more technical terms, a spacetime is stationary if it admits a timelike Killing vector $\xi^{\alpha} = \partial_t$. A Killing vector obeys the following equation, where the brackets imply summation over the permutations

$$\nabla_{(\mu}\xi_{\nu)} = 0 \tag{2.1}$$

A timelike killing vector obeys

$$g_{\mu\nu}\xi^{\mu}\xi^{\nu} < 0 \tag{2.2}$$

A spacelike Killing vector χ^{μ} obeys

$$g_{\mu\nu}\chi^{\mu}\chi^{\nu} > 0 \tag{2.3}$$

A Killing vector represents a certain symmetry in the metric. For example, the timelike killing vector represents a symmetry under time translation. We later use the presence of Killing vectors to calculate conserved quantities. All the spacetimes we will consider during this thesis will be stationary ones. For a timelike Killing vector the conserved quantity is energy, while for a spacelike Killing vector the conserved quantity is a momentum.

2.2 Schwarzschild Solution

In order to derive the Schwarzschild metric we will start from assuming the existence of a spherically symmetric solution. Let us consider the implications of solutions with a spherical symmetry. The angular components of the metric must be equivalent to the surface of S^2. This would imply

$$g_{\phi\phi} = sin^2(\theta) g_{\theta\theta} \tag{2.4}$$

For a spherically symmetric metric, we can multiply the angular components of the metric or any other component with an arbitrary function in r without affecting the spherical symmetry of the solution. The metric tensor would then take the form

$$\begin{bmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & C(r) & 0 \\ 0 & 0 & 0 & C(r)sin^2(\theta) \end{bmatrix} \tag{2.5}$$

We can now define a new radial coordinate such that

$$\bar{r}^2 = C(r) \tag{2.6}$$

For the functions $A(r)$ and $B(r)$ it is easy to relabel them in terms of $\bar{r}$ so that they become $A(\bar{r})$, and $B(\bar{r})$. This ensures that our metric components are all functions in the new coordinate $\bar{r}$. Having done this, we can now drop the bar and simply write r since it is only a label as of now.

We now bring out the field equations. We are interested in the region outside the spherically symmetric mass distribution. This would correspond to the vanishing of the Ricci tensor.

$$R_{\mu\nu} = 0 \tag{2.7}$$

This gives us a set of differential equations to solve. The solution to these differential equations gives us B and A.

$$\begin{aligned} B &= \left(1 - \frac{D}{r}\right)^{-1} \\ A &= c\left(1 - \frac{D}{r}\right) \end{aligned} \tag{2.8}$$

Under a re-scaling of the time coordinate $t \implies tc^{-1}$ we get the following metric where D is a constant of integration.

$$\begin{bmatrix} -(1-\frac{D}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{D}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{bmatrix} \tag{2.9}$$

To find d we will need to examine the classical limit. By taking the classical limit we mean that all velocities are negligible with respect to the speed of light, while the gravitational field is both static and weak. We will do so by examining a geodesic in the classical limit. The metric would be a slight perturbation to the Minkowski metric such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.10}$$

Let us consider an affinely parameterised geodesic

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.11}$$

At low velocities we can see that the change along the geodesic of the spacial components is relatively negligible

$$\frac{dx^0}{d\lambda} \gg \frac{dx^i}{d\lambda} \tag{2.12}$$

We also write out the Christoffel's symbol to highlight the simplification that occurs for stationary spacetimes.

$$\begin{aligned} \Gamma^\alpha_{00} &= \frac{1}{2}g^{\alpha\mu}(\partial_0 g_{\mu 0} + \partial_0 g_{0\mu} - \partial_\mu g_{00}) \\ &= -\frac{1}{2}g^{\alpha\mu}\partial_\mu g_{00} \\ &= -\frac{1}{2}\eta^{\alpha\mu}\partial_\mu h_{00} \end{aligned} \tag{2.13}$$

We can then take note of the fact that the perturbation must be time independent, too.

$$\partial_0 h_{00} = 0 \tag{2.14}$$

The geodesic equation for $\alpha = 0$ then becomes:

$$\frac{d^2x^0}{d\lambda^2} = 0 \tag{2.15}$$

Let's examine the case where $\alpha = i$ is one of the spacelike coordinates.

$$\frac{d^2x^i}{d\lambda^2} = \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{d\lambda}\right)^2 \tag{2.16}$$

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \tag{2.17}$$

Let us recall the acceleration due to a Newtonian gravitational field where Φ is the gravitational potential

$$\begin{aligned} a &= -\nabla\Phi \\ \Phi &= -\frac{m}{r} \end{aligned} \tag{2.18}$$

Comparing both equations we arrive at

$$\begin{aligned} h_{00} &= -2\Phi \\ g_{00} &= -(1 + 2\Phi) \\ &= -\left(1 - \frac{2m}{r}\right) \end{aligned} \tag{2.19}$$

Comparing this with our metric timelike components it is straightforward to see that

$$D = -2m \tag{2.20}$$

We assumed several things during our derivation of the Schwarzschild spacetime. Namely, we assumed that it is both isotropic and static. What would have happened had we only assumed isotropy? What would a time-dependent spherically symmetric solution look like? Birkhoff's theorem[4] states that the Schwarzschild solution is the only spherically symmetric vacuum solution and that there can be no time dependent spherically symmetric vacuum solution.

For the case of the Schwarzschild metric, the event horizon is straight forward and only one exists. Let us examine what happens as

$$r = 2m \tag{2.21}$$

The g_{rr} in the metric becomes divergent and the timelike component vanishes completely. As it decreases beyond that point, the radial and time coordinates exchange signs. The radial coordinate becomes timelike while the time coordinate becomes spacelike.

What is important to note is that there is a Killing horizon along the event horizon. This is the case for the Schwarzchild metric. The Killing horizon occurs when $g_{tt} = 0$. The event horizon is situated at $g^{rr} = 0$, and corresponds to an infinite red shift. Let us first define what a Killing horizon is.

A Killing horizon is where a Killing vector field is null along some null hypersurface. In the Schwarzschild case we can see

$$\xi_\alpha \Big|_{r=2m} = 0 \tag{2.22}$$

One thing we have not discussed is the presence of curvature singularities. The singularity in the metric is a coordinate one, and can be fixed by an appropriate coordinate transformation. Some singularities, however, cannot be avoided. To find this, we rely on curvature invariants. For example, the contraction of the Riemann tensor with itself is one of them and gives.

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48m^2}{r^6} \tag{2.23}$$

As we can see above, it diverges as r approaches zero. Unlike the singularity in the metric, this is a physical one, and cannot be avoided through a coordinate transformation.

2.3 Reissner-Nordström Solution

The Reissner-Nordström Solution is a static, charged exact solution to the Einstein field equations. This solution differs from the Schwarzschild one because it is not electrically neutral. Thus, we would need to consider the effect on the metric and on the equations of motion. Whereas the energy-momentum tensor was null outside the horizon in the Schwarzschild case, it will not be for the sner-Nordström one. This is because of the presence of electromagnetic sources. When we have sources the solution is not a vacuum one.

We will be using a heuristic argument that will extend the techniques used in solving for the Schwarzschild spacetime. One important thing to note is that while we considered Newtonian gravity to be the limiting case before, we will now consider the Schwarzschild metric to be the limiting case of our solution. When the charges die out, we recover the Schwarzschild spacetime.

We will be considering the electromagnetic field strength tensor $F_{\mu\nu}$ in our solution. The tensor represents the electric and magnetic fields in the 3 spatial dimensions. The tensor is antisymmetric, meaning

$$F_{\mu\nu} = -F_{\nu\mu} \tag{2.24}$$

The only components that will survive are the ones associated with the radial magnetic and electric fields. This means that all the terms are identically zero except for 4 terms. Since the tensor is antisymmetric, this would leave us with two independent components. The ones associated with the radial electric and magnetic field are

$$\begin{aligned} F_{tr} &= E_r \\ F_{\theta\phi} &= -B_r sin(\theta) \end{aligned} \tag{2.25}$$

The $sin(\theta)$ in the expression above has to do with the radial component of a the magnetic field for spherical symmetry. It goes as $sin(\theta)^{-1}$

Through solving the equations and taking the classical limits. We would expect the fields to be as they are in flat spacetime. Building on this, we find

$$\begin{aligned} E_r &= \frac{q}{r^2} \\ B_r &= \frac{p}{r^2} \end{aligned} \tag{2.26}$$

Where q is the electric charge of the black hole and p its magnetic charge.

We examine the energy-momentum tensor and see how it is affected by the field strength tensor. As opposed to the Schwarzschild case, the energy momentum tensor will not vanish everywhere. The energy-momentum tensor is

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{2.27}$$

We solve for an extra terms, while maintaining the assumptions made about the components of $F_{\mu\nu}$ the solution to the metric becomes straightforward.

The line element for the sner-Nordström spacetime is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{2.28}$$

Where

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2 + p^2}{r^2} = \frac{r^2 - 2mr + q^2 + p^2}{r^2} \tag{2.29}$$

The event horizon occurs at

$$g^{rr} = f(r) = 0 \tag{2.30}$$

There is no longer an event horizon at $r = 2m$ as was the case in the Schwarzschild solution. The electromagnetic charges have shifted the event horizon. Looking at the rightmost part of the above equality, we will see that there are two distinct horizons. This is because of the r^2 dependence in the numerator that coincided with the addition of the electromagnetic terms to the metric. The horizons are situated at

$$r_{\pm} = m \pm \sqrt{m^2 - (q^2 + p^2)} \tag{2.31}$$

As we will do later, it is always important to note that we should return to the limiting case whenever our additional variables are set to zero. The equation above shows that we only get one physical radius at $r = 2m$.

Examining the event horizon radius, we would get three distinct possibilities depending on the relative values of m, q, and p. The square root will either exist, be zero, or be imaginary. We will consider each case on its own. As was the case in the Schwarzschild solution, there is a singularity that exists at $r = 0$ We can see that by a calculation of $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ as we did before. Before, our singularity was well behaved, hidden behind the event horizon. Not all singularities afford us that courtesy.

The first case

$$m^2 < q^2 + p^2 \tag{2.32}$$

In this case, the radii will both be imaginary. This means that there will be no event horizon at all. For all points in this spacetime, $g_{tt} \neq 0$, which means that the time coordinate always remains timelike, while the radial coordinate is always spacelike. The singularity that exists at $r = 0$ then becomes a ”naked” singularity. Which means that we do not have a horizon to hide the singularity behind.

Penrose, in introducing the weak cosmic censorship hypothesis[5] rejected the existence of naked singularities. The hypothesis reasons that with a naked singularity, causality may be broken. This takes away the predictivity of physics.

Due to the above reasons, there is hardly any interest in this solution, since it is unphysical.

The second case

$$m^2 = q^2 + p^2 \tag{2.33}$$

In this case there is an event horizon, namely at $r = m$. It is called the extreme sner-Nordström solution. Needless to say, the solution is highly unstable under perturbations, any increase in mass or charge will change it into one of the other two cases.

Our primary case of interest, and the more physical case is the next one. It is a solution where the horizon is stable under perturbations and where no naked singularity exists.

The third case

$$m^2 > q^2 + p^2 \tag{2.34}$$

Here, there are two event horizons, r_+ and r_-. However, the only event horizon that can be ”seen” by an observer at infinity is the larger one. As with the Schwarzschild case, the time coordinate becomes spacelike beyond the outer horizon while the radial coordinate becomes timelike. Contrastingly, they switch back again after crossing the second horizon.

The radius of interest to us will be the larger of the two. The sner-Nordström solution is not the only case where there is more than one horizon for a the black hole. Sometimes the equation is solvable in terms of r. Herein, whenever there is more than one radius, we will refer to the largest one r_+ as the horizon radius r_h. In more complicated cases there is no analytic solution for r at all, and we will have to find some other way to parameterise.

2.4 GR with a Cosmological Constant Λ

A constant value multiplied by the metric, is the most trivial possible addition to the Einstein field equations. This constant Λ is commonly known as the cosmological constant. Einstein had once called it his life's greatest blunder. This was due to his belief in Mach's principle that dictated that the universe must be static.[6] We show the field equations with Λ included below

$$R_{\mu\nu} + g_{\mu\nu}\left(\Lambda - \frac{1}{2}R\right) = T_{\mu\nu} \tag{2.35}$$

It is only in the 90s, that there was a vast reignited interest in it. This came as a result of observations that showed that the expansion of the universe is accelerating. This accelerated expansion could be explained through a positive cosmological constant. A spacetime with a positive constant is call a de Sitter spacetime, *dS* for short.

Another development in the 90s that brought great attention to the cosmological constant [7] was the AdS-CFT correspondence. AdS is short for Anti de Sitter, it is used to refer to spacetimes where the cosmological constant is negative. CFT stands for conformal field theories, they are a class of quantum field theories that are invariant under conformal transformations. Loosely, this means that the theory is invariant under a change in scale.
The technicalities of the correspondence are beyond our scope in this thesis. However, we will note the cause for the interest in it. The secret lies in the correspondence between two very different field theories. To be more specific, the correspondence is an equivalence between the boundary of $d + 1$-dimensional AdS spacetimes and conformal field theories of dimension d. It is regarded by many as one of our best chances at uncovering a theory of quantum gravity.

The sign of the scalar Λ represents whether the spacetime is positively or negatively curved. It's important to note that the cosmological constant is directly related to the dimension of the spacetime itself. The l^2 below is called the AdS radius. This is given by the relation

$$\Lambda = -\frac{(d-2)(d-1)}{2l^2} \tag{2.36}$$

A pure AdS spacetime will have a metric that takes the following form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + sin(\theta)^2 d\phi^2\right) \tag{2.37}$$

Where

$$f(r) = 1 + \frac{r^2}{l^2} \tag{2.38}$$

We will go into more detail as to the effects of the cosmological constant when we consider the Taub-NUT-AdS metric itself.

Chapter 3

Black Hole Thermodynamics

In this chapter we will be discussing classical thermodynamic systems and phase transitions, focusing on the Van der Waals fluid phase transitions. It's important to draw an analogy between our phase transitions and those that occur in Van der Waals fluids. Later in the chapter, we examine the first law, the Smarr's relation and the Gibbs-Duhem relation in each case.

3.1 Classical Thermodynamics

We will begin with a statement of a combination of the first and second laws of thermodynamics. The first law is a statement about the conservation of energy while the second is concerned with the reversibility of processes. However, we will be referring to the the form below as the first law within this thesis. Arguably, the first noting of energy conservation was by Leibniz in 1693[8] where he stated that the total energy for a point particle in a gravitational field is conserved. The principle of energy conservation is at the core of scientific thought and reasoning. Below, we present the first law.

$$dU = TdS - PdV + \sum_{i=1}^{N} \mu_i dN_i \tag{3.1}$$

In the equation above, dU is the change in the internal energy of the system. S is the entropy while T is the temperature, the term represents the heat transfer between the system and its surroundings. P is the pressure and V is the volume, this term represents the amount of work done by the system on its surroundings. In the last term, μ_i is an arbitrary chemical potential while N is the amount of charge associated with the potential.

As we can see in the first law, each parameter has a conjugate thermodynamic quantity. One of these quantities would be an intensive parameter while the other would be extensive. For example, temperature is an intensive parameter while entropy is extensive. Let us consider two systems of identical gases at the same pressure and volume with an infinitesimally thin diathermal wall between them. Let us further suppose that we know a magician that can magically make the wall vanish completely.

What happens when it does? The temperature would stay the same, as well as the pressure these are intensive parameters. What would happen to the volume? How would the entropy change? They would increase with the scale of the system, they are extensive parameters. This is what separates the extensive and intensive parameters. The extensive parameters depend on the *extent* of the system whereas intensive ones depends on the *internal* state of the system. It is trivial to note that the *magnitudes* of chemical potentials are intensive while the *amount* of charges is extensive.

We will now consider the Gibbs-Duhem relation. The Gibbs-Duhem relation can be derived from the first-order homogeneity of the first law. Perhaps we should bring back our example but make the size of both systems arbitrary. Let us first consider one of them, noting that after joining both systems, $V' = \lambda V$. What then happens to the rest of the system? S would scale in the same manner as V which would then result in λdU. We can use this to apply the Euler homogeneous function theorem, leading to

$$U(S,V,N) = TS - PV + \sum_{i=1}^{N} \mu_i N_i \tag{3.2}$$

Let us now take the differential of the equation.

$$dU = TdS + SdT - PdV - VdP + \sum_{i=1}^{N} \mu_i \, dN_i + \sum_{i=1}^{N} N_i \, d\mu_i \tag{3.3}$$

Through subtracting the first law from the equation above we then get the Gibbs-Duhem relation.

$$0 = SdT - VdP + \sum_{i=1}^{N} N_i \, d\mu_i \tag{3.4}$$

It is important to note that there are two different types of distinct thermodynamic systems. Ones that are described by a canonical ensemble, and others that require the grand canonical ensemble. The key difference is that for the canonical ensemble, the number of charges are fixed. For the grand canonical ensemble, however, it is the potentials that are fixed. This allows for the number of charged particles to be exchanged in the case of the grand canonical ensemble, while it is not allowed to change for canonical ensemble.

We will now introduce two different thermodynamic potentials and the physical properties that are associated with each of them. The potentials are the Helmholtz and Gibbs free energies, both deal with the stability of a thermodynamic system, but the Helmholtz energy does not consider the change in charges while the Gibbs energy does. The Helmholtz energy F and the Gibbs energy G. As differentials, they are given by the following equations

$$\begin{aligned} dF &= -SdT - PdV + \sum_{i=1}^{N} N_i \, d\mu_i \\ dG &= -SdT + VdP + \sum_{i=1}^{N} \mu_i \, dN_i \end{aligned} \tag{3.5}$$

The Helmholtz energy is related to the canonical ensemble by the following relation

$$Z_{canonical} = e^{-\beta F} \tag{3.6}$$

Where

$$\beta = \frac{1}{k_b T}. \tag{3.7}$$

Generally speaking, we can write

$$Z_\alpha = e^{-\beta\Omega} = e^{-I} \tag{3.8}$$

Where I is the Lagrangian action and Ω is an arbitrary potential depending on the type of system. This is important to note because some systems, like the one we will consider in this thesis, are not entirely canonical nor grand canonical. They are mixed systems where for some conjugate quantities the potential is fixed while the charge is fixed for others.

3.2 Classical Phase Stability and Transitions

In this sub-chapter we will establish the properties of a physical phase. We pay special attention to the first order phase transitions for Van der Waals fluids, and the critical behaviour they show. The main reason is that similar behaviour exists in black hole thermodynamics when we allow the cosmological constant to vary. This analogy was first discussed by Kubiznak and Mann.[9]

Let us first recall the ideal gas molar equation of state.

$$Pv = RT \tag{3.9}$$

The Van der Waals fluid equation can be reached by adding two assumptions to ideal gas equation of state.[8] The first is that the molecules are not point particles but do take up some of the volume of the container. The second one relies on interaction between the particles and themselves. A particle nearing the wall of the container will experience inter-molecular forces on one side only when they are colliding with the container wall. This decreases the amount of force they exert on the container. By applying these assumptions where a and b are empirical constants we arrive at the following equation of state.

$$P = \frac{RT}{v-b} - \frac{a}{v^2} \tag{3.10}$$

We will first consider what we mean by a physical or metastable phase. We identify them through considering two different properties. The first one we look at is the condition for thermal stability, the heat capacity. The second is concerned with mechanical stability and is the isothermal compressibility.

The heat capacity while keeping the pressure constant is the the amount of energy needed to raise the temperature of a system by a certain amount. The heat capacity at constant pressure for an arbitrary mechanical system is given by

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P \tag{3.11}$$

The compressibility has to do with the instantaneous volumetric response of a system when pressure on it is varied. We can find it using the following equation.

$$\kappa = \frac{-1}{V}\left(\frac{\partial V}{\partial P}\right)_S \tag{3.12}$$

Now, let us consider phase transitions in general. A second order phase transition occurs when there is a continuous phase. One case where this takes place is for super-critical fluids; fluids at pressures and temperatures above the critical point. For a first order phase transition, there are two distinct phases and a discontinuity in the entropy. An example of a first order phase transition is the transition between water and steam where a latent heat is involved.

The separating point between a first order phase transition and a second order phase transition is the critical point. On one side of the critical point a first order transition takes place while on the other there is a second order one. To solve for the critical point, we need to solve the following set of equations

$$\frac{\partial P}{\partial v} = 0 = \frac{\partial^2 P}{\partial v^2} \tag{3.13}$$

For a Van der Waals fluid this would be given by

$$\frac{-RT}{(v-b)^2} + \frac{2a}{v^3} = 0 = \frac{2RT}{(v-b)^3} + \frac{6a}{v^4} \tag{3.14}$$

We can then solve for a critical volume v and temperature T, doing so for $a = 3$ and $b = 5$. Plotted below, is the variation of the pressure with the volume at various isotherms surrounding the critical temperature.

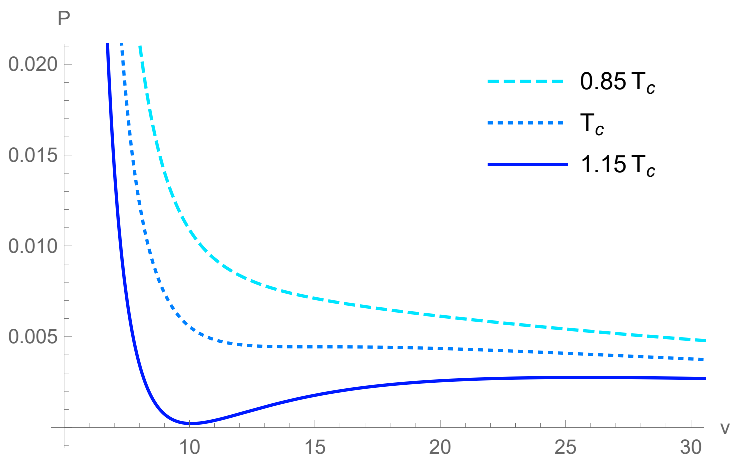

Figure 3.1: The pressure versus the volume as the temperature varies around the critical point for Van der Waals fluid with $a = 3$ and $b = 5$

As we can see in figure 3.1, the behaviour of the pressure versus the volume differs around the critical temperature. Below the critical point the pressure is not one-to-one, there is a region with positive slope. In that region, the compressibility is not positive, and the curve does not represent a physical phase.

This nonphysical region of the isotherm is then replaced by an isobar in accordance with the Maxwell equal area law. The law states that we can use a horizontal line across the positive slope region to find the physical pressure and temperature for the transition. The horizontal line satisfies the equal area law if the areas bounded between it and the curve above and below the line are equal. This signifies that the temperature and pressure of both points are equal since they lie on the same isotherm and happen at a constant pressure.

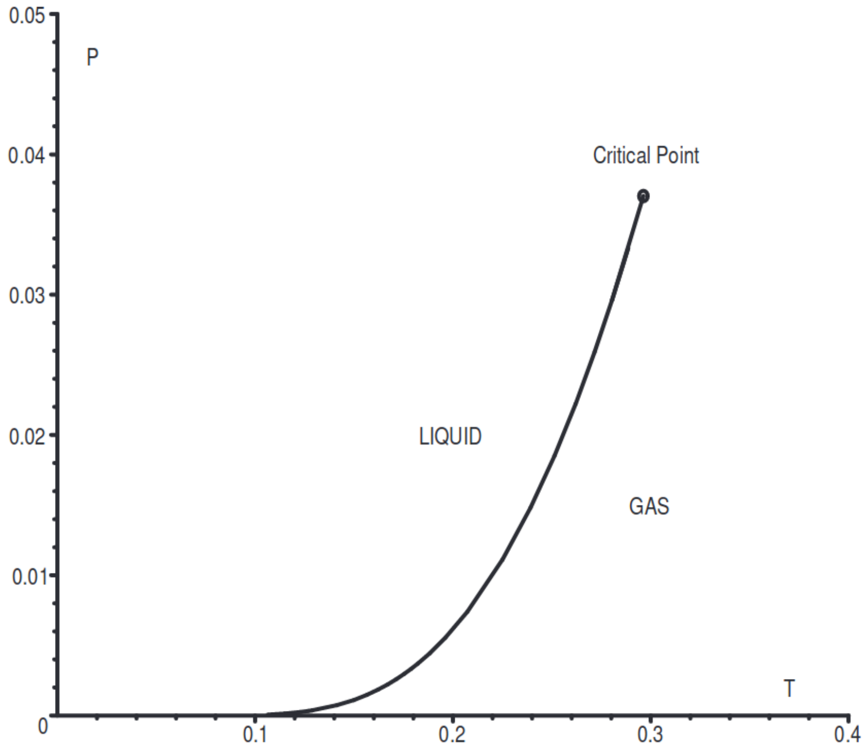

Figure 3.2: Van der Waals phase diagram[9]

The fluid undergoes a first order phase transition below the critical temperature. At the critical temperature, and higher temperatures, the phase transition is second order and there are no discontinuities in the entropy. It's important to note that this is only an example. As we will see, it is possible for more than one critical point to exist. In the case of the Van der Waals fluid, this is entirely dependent on the values of a and b. We will have a similar dependence on a constant in our thermodynamic phase structure.

In the phase diagram above, there is a first order transition across the line. For the fluid to change its phase through crossing the line, there must be a latent heat involved. This is not the case above the critical point and the transition is continuous. Across the line there is a discontinuity in the entropy. In the region above the critical point, there is none.

3.3 Black Holes from Mechanics to Thermodynamics

Initially, Black hole thermodynamics was not an established field. There were two separate fields. One of thermodynamics, and one of black hole mechanics. Black hole mechanics were a set of laws that all black holes adhered to. It is only after the identification made by Bekenstein [10] of the entropy with a fourth of the area of the horizon, that the fields truly met.

Bekenstein first appealed to a result that was found by Hawking [11] in 1971. Hawking had proved that the area of a black hole cannot decrease in any process. This drove Bekenstein to propose a correspondence between the black hole surface area, and its entropy.

He then proposed a generalised second law that states ”the common entropy in the black-hole exterior plus the black-hole entropy never decreases.” [10] Even then, in the paper, he made sure to state that the conjugate thermodynamic variable to entropy was not a physical temperature.

It was Hawking who first suggested that the black hole temperature could be a physical phenomena. [12] Hawking suggested that the event horizon of a black hole could emit particles. He also presented a correspondence between the physical temperature and the surface gravity of the black hole.

In this thesis we will be working in a regime that has come to be known as *extended* thermodynamics. This is different from classical black hole thermodynamics. The difference is that we allow the cosmological constant to vary. While doing so, we define a correspondence between it and the pressure. Before that black holes were seen as systems that can do work either through their angular momentum J or electromagnetic interactions only. As presented in [2] for the Kerr metric this took the form

$$dM = \frac{\kappa}{8\pi}dS + \Omega_H dJ \tag{3.15}$$

The analogue to the zeroth law, is the statement that surface gravity κ is constant along the event horizon. This is the case at least in the situations where the killing and event horizons coincide. [2]

How would we approach a problem that is more complex than the Kerr solution? Especially one where it is not as easy to set apart the different thermodynamic parameters. As with any thermodynamic system we can calculate an action, a partition function, and a corresponding chemical potential.

Gibbons and Hawking [13] considered the path integral method of quantizing gravity. However, they found that the manifold the were integrating on had a conical singularity. The trick they used was to complexify the metric and then calculate the action on a contour that avoids the singularity. The partition function they used was

$$Z = \int d[g]d[\phi]e^{iI[g,\phi]} \tag{3.16}$$

The RHS is broken down as follows, $d[g]$ is a measure on the space of metrics, $d[\phi]$ on the space of matter fields. The action $I[g,\phi]$ is dependent on both. The intricacies of the path integral formulation itself are beyond the scope of this thesis, and we will be treating it as such.

The action they considered was comprised of the Einstein-Hilbert (EH) action, and a boundary term they introduced. The EH action is an integral of the Ricci scalar R over the bulk of the spacetime. The integral over the boundary has come to be known as the Gibbons-Hawking (GH) boundary term.

The need for the boundary term arose from the nature of the Ricci scalar. The path integral method requires that we deal with the metric, and its first derivatives only. However, R is also a function of the second derivatives of the metric. Through an integration by parts, we can be relieved of this behaviour. This results in the Gibbons-Hawking boundary term.

Below, we can see the form for both action terms

$$
\begin{aligned}
I_{EH} &= \frac{-1}{16\pi}\int_M d^4x\sqrt{|g|}R \\
I_{GH} &= \frac{-1}{8\pi}\int_{\partial M} d^3x\sqrt{|h|}\,K
\end{aligned}
\tag{3.17}
$$

Above, $\sqrt{|h|}$ is the root of the determinant of the boundary metric. The boundary metric depends on what kind of boundary we will be considering. For our purposes, the boundary is taken to be an r-normal surface that we then evaluate in the limit where r tends to infinity. K represents the extrinsic curvature of the boundary.

To get the boundary metric from the spacetime metric we subtract from the metric the normal vector in r. For a spacetime where $g_{r\mu} = g_{rr}\delta^r_\mu$ this takes the form

$$
\begin{aligned}
\hat{n}_r &= [0, \sqrt{g_{rr}}, 0, 0] \\
h_{\mu\nu} &= g_{\mu\nu} - \hat{n}_r\hat{n}_r
\end{aligned}
\tag{3.18}
$$

To get the extrinsic curvature one may use

$$
K = h^{\mu\nu}\nabla_\nu\hat{n}_\mu \tag{3.19}
$$

The action above, as it is, diverges. There is more than one method to deal with this problem. We will discuss both. However, we only focus on the method we will use during this thesis.

The first method is to use background subtraction. This works through taking the action of the black hole metric and then subtracting from it the background spacetime. The second way, which we will adopt, is the counterterm method. This term relies on terms inherent to the boundary to cancel the divergences.

The background method is tricky and could have several pitfalls. First of all, we need to make a decision on what the background is. For the case of Taub-NUT-AdS for example, this would lead to some ambiguity that we could do without. We also need to consider that some information about the behaviour at the boundary could be lost during the subtraction procedure. This isn't the best prospect within the context of AdS-CFT.

The counterterm method, in contrast, does not rely on the matching of a boundary.

Since it is only on the boundary, there are no worries about the equations of motion being affected. It also leaves gives us the luxury of not losing information about the boundary. In addition to all of that, the method is systematic. Its limitations mainly lie in its constriction to AdS spacetimes. The counter terms for a general AdS spacetime were first presented in [14], for AdS_4 we get

$$I_{CT} = \frac{-1}{4\pi l} \int_{\partial M} d^3x \sqrt{|h|} \Big(1 - \frac{l^2}{4} R \Big) \tag{3.20}$$

The action counterterm for Taub-NUT-AdS_4 differs slightly and was calculated in [15]

$$I_{CT} = \frac{1}{4\pi l} \int_{\partial M} d^3x \sqrt{|h|} \Big(1 + \frac{l^2}{4} R \Big) \tag{3.21}$$

In our work, we will not be using the Einstein-Hilbert action for the bulk. This is due to some additions that we must make on the account of having a nonzero cosmological constant, as well as electromagnetic charges. The action we will use will be the Einstein-Hilbert-Maxwell one, and it takes the following form

$$I_{EHM} = \frac{-1}{16\pi} \int_M d^4x \sqrt{|g|} \Big(R + \frac{6}{l^2} - F^2 \Big) \tag{3.22}$$

This brings us to the total action we will consider

$$I = I_{\text{EHM}} + I_{\text{GH}} + I_{\text{CT}} \tag{3.23}$$

3.4 Phase Transitions

In this sub-chapter we will be sampling some of the black hole phase transitions from the literature. We begin with considering one of the first phase transitions that were discovered in black hole thermodynamics. This will be the Hawking-Page phase transition.

The Hawking page phase transition occurs in Schwarzchild-AdS black holes. We will calculate the action using the counter term method. We will then look at the type of phase transition that occurs. The action is given by

$$I = I_{\text{EH}} + I_{\text{GH}} + I_{\text{CT}} \tag{3.24}$$

We drop the M from the Einstein-Hilbert-Maxwell term as the solution is uncharged.

This means F^2 will be zero. Upon evaluating the action we find it to be

$$I = \frac{\beta}{2}\Big(m - \frac{r_h^3}{l^2}\Big) \tag{3.25}$$

This result is exactly the result we would get if we were to follow a background subtraction method. In this case we would have subtracted a pure AdS background from the action. β above is the reciprocal of the temperature where the black hole temperature is equal to

$$T = \frac{3r_h^2 + l^2}{4r_h\pi l^2} \tag{3.26}$$

We can calculate the entropy directly from the action by using the following relation

$$S = \beta\partial_\beta I - I \tag{3.27}$$

This is a quarter of the horizon surface areas as expected

$$S = \pi r_h^2 \tag{3.28}$$

We can define a heat capacity for the black hole. This helps us quickly identify the stable from the unstable phases. Generally speaking, for an arbitrary parameter j kept constant, we can calculate a heat capacity c_j through

$$c_j = T\Big(\frac{\partial S}{\partial T}\Big)_j \tag{3.29}$$

The j around the bracket indicates that we are taking the derivative whilst keeping the parameter j constant. One observation that will helps us is that the entropy is a monotonic function in r. This allows us to use the chain rule to write

$$c_j = T\Big(\frac{\partial S}{\partial r_h}\Big)_j\Big(\frac{\partial r_h}{\partial T}\Big)_j \tag{3.30}$$

This means that we only need to consider regions where the value of the partial derivative of r with respect to T is positive. Better yet, the reciprocal $\frac{T}{r_h}$ always shares the same sign. This allows us to tell which radii have a positive heat capacity by examining the slope of the T vs r_h graph.

For a Hawking-Page transition we do not think of the pressure as a dynamic variable. It is a constant. Thus, it is useful to plot the variation of T with respect to r_h in an l

independent form. It shows up here as a constant scaling factor that does not change the behaviour of the graph. This can be reached by plotting $T \cdot l$ versus r_h/l.

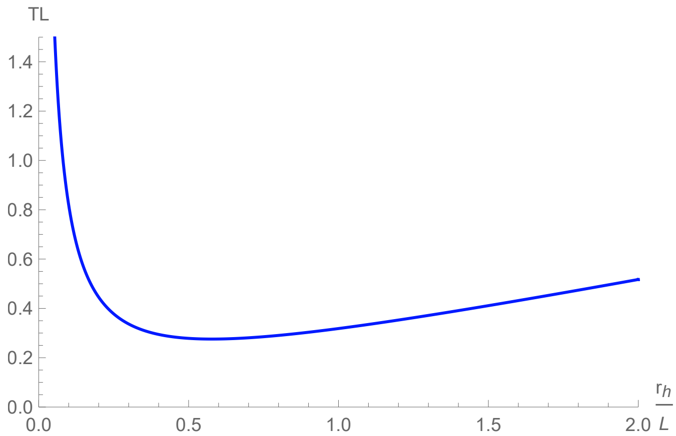

Figure 3.3: Hawking-Page phase transition

The graph above sets a minimum temperature for the existence of a black hole. Below the minimum T, no black hole solution can exist. Above it two radii exist for each temperature, but only one is stable.

Before the minimum point in the graph, the black hole exhibits a negative heat capacity. This shows us that it is not a stable phase. After the minima, however, the temperature rises with the radius, indicating a stable phase. To calculate the minimum point we only need to take the first derivative in T and set it equal to zero. This results in a radius

$$r_{h,min} = \frac{l}{\sqrt{3}} \tag{3.31}$$

The Helmholtz free energy F can be calculated through dividing the action with β. We can use the free energy to find the point where the black hole becomes the favorable to the background. This gives us

$$r_c = l \tag{3.32}$$

This means that for radii that are greater than l the black hole will be preferable to the AdS background. This means that between r_{min} and r_c there is a region where the background is preferable even though the phase is stable. For radii below r_{min} the black hole is unstable.

The analysis we have done thus far, is that of ”normal” phase space. We will examine the thermodynamics of the sner-Nordström-AdS solution while allowing the cosmological constant to vary. This will be analysis that is carried out in extended phase space. We will discuss conditions on stability that relate to both the heat capacity and the compressibility. As was the case with the heat capacity, negative compressibility implies an unstable phase.

The action we consider here will take into account F^2 since we have an electric charge. For a spherically symmetric solution, the existence of a magnetic charge will not affect the behaviour. Thus, we will only consider and electric charge, q. The one-form A_μ then becomes

$$A_\mu = \left[\frac{-q}{r} + \phi, 0, 0, 0\right] \tag{3.33}$$

while

$$f(r) = \frac{(r^2 - 2mr + q^2)l^2 + r^4}{r^2 l^2} \tag{3.34}$$

Upon calculating the action, we find

$$I = \frac{\beta\Big(l^2(mr_h - q^2) - r_h^4\Big)}{2r_h l^2} \tag{3.35}$$

The mass and temperature take the following forms

$$\begin{aligned} m &= \frac{(r_h^2 + q^2)l^2 + r_h^4}{2r_h l^2} \\ T &= \frac{3r_h^2 + l^2(1 - \phi^2)}{4r_h \pi l^2} \end{aligned} \tag{3.36}$$

To examine any of the thermodynamic relations we need to write them in terms of parameters that we know are fixed for the action integral. Recall that A_t is the scalar potential, thus by fixing it we choose to fix the electric potential instead of the charge. This makes us work in the grand canonical ensemble. It would be the canonical ensemble if we fix the charges instead of the potential. We substitute each q and each l in accordance with

$$\begin{aligned} P &= \frac{3\pi}{8l^2} \\ \phi &= \frac{q}{r_h} \end{aligned} \tag{3.37}$$

The temperature becomes

$$T = \frac{8\pi r_h^2 P + 1 - \phi^2}{4r_h\pi} \tag{3.38}$$

We can rearrange it to reach an equation for the pressure.

$$P = \frac{T}{2r_h} + \frac{\phi^2 - 1}{8r_h^2\pi} \tag{3.39}$$

As r approaches zero, there are three possibilities. Either $\phi^2 > 1, \phi^2 < 1$, or $\phi^2 = 1$. For $\phi^2 = 1$ we get completely linear behaviour in T, and a gradient that is positive everywhere. The gradient is also everywhere positive for the case for the case where $\phi^2 > 1$. We can see Hawking-Page-like behaviour whenever $\phi^2 < 1$, but none otherwise. Taking P as a thermodynamic parameter we can plot the T vs r graph at a constant pressure.

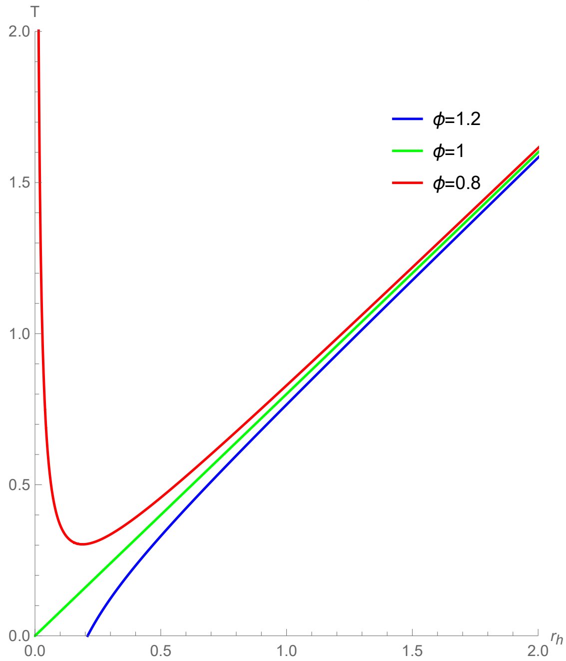

Figure 3.4: As the electric potential changes, the behaviour of the black hole changes as well.
Plotted for P=0.4

Now let us turn our attention towards the pressure. We first need to recall the definition for compressibility. The compressibility κ is used to measure the instantaneous change in volume relative to a change in pressure. It is expressed as

$$\kappa = \frac{-1}{V}\left(\frac{\partial P}{\partial V}\right) \tag{3.40}$$

A negative compressibility corresponds to a system that is not stable as was the case with the heat capacity. Just as we were able to tell if a particular range of radii was stable or unstable through examining the slope of the temperature with respect to r_h, we are also able to do so for the pressure. The difference is that for a positive compressibility, we the slope should be negative.

As r_h tend towards infinity, it approaches zero from the positive side. This means that for a solution to be stable everywhere, we would need it to approach positive infinity. As such, when $\phi^2 < 1$ we get an unstable phase at low radii. There is little difference between the graphs for $\phi^2 = 1$ and $\phi^2 > 1$ as both follow very similar behaviour.

While there is minimum temperature where a black hole can exist, for the pressure it is the complete opposite. As shown in figure 3.4, for a black hole where $\phi^2 < 1$ the pressure approaches negative infinity for small values of r_h. Firstly, this places a limit for the minimum stable black hole radius that could exist. It also places a limit on the pressures where a black hole can exist. Beyond a certain pressure, no black holes can exist, not even unstable ones.

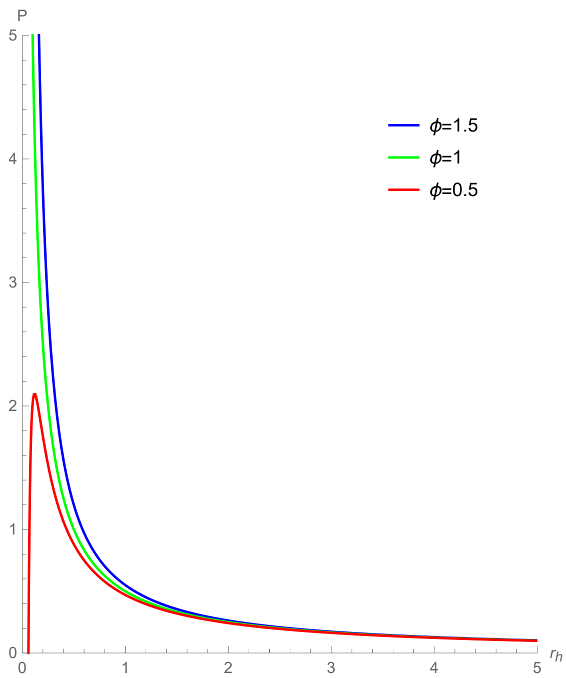

Figure 3.5: The change in pressure with respect to the radius at constant temperature at different electric potential values.
Plotted for T=1

The last thing we will consider is the grand canonical potential, we need to make sure that our thermodynamics are consistent function. This would entail several relations. Some of these are between the thermodynamic variables and themselves, while others are related to the grand potential. We expect it to take the form

$$d\Omega = -SdT + VdP - qd\Phi \tag{3.41}$$

By taking the partial derivatives we can show that these correspond to the right quantities, namely

$$\begin{aligned} -\left(\frac{\partial \Omega}{\partial T}\right)_{P,\Phi} &= S = \pi r_h^2 \\ \left(\frac{\partial \Omega}{\partial P}\right)_{T,\Phi} &= V = \frac{4\pi r_h^3}{3} \\ -\left(\frac{\partial \Omega}{\partial \Phi}\right)_{T,P} &= q = \phi r_h \end{aligned} \tag{3.42}$$

One of the very important differentiating aspects between thermodynamics in general and black hole thermodynamics is the dependence on length. In a regular classical system, an ideal gas for example, the energy and mass are volume-dependent quantities. For a black hole the mass scales as a length. This leads us to our next test for our formulation.

An important relation to consider is the Smarr formula.[16] The Smarr formula traditionally relates the dimensionality of the constituents of the mass to each other. Without allowing the cosmological constant to vary, it would have been the first law, as the mass would represent the internal energy. However, in the extended phase space the mass represents the enthalpy of the spacetime.

Let us consider our case. We know that the mass is associated with the enthalpy for extended phase spaces. To reach the mass from the grand potential, we apply a legendre transform

$$M = \Omega + TS + q\phi \tag{3.43}$$

It is comforting that the above relation holds, however, it is not what we are trying to prove. What we are interested in is how the differential for M would behave. After the transform we should get

$$dM = TdS + VdP + \phi dq \qquad (3.44)$$

The Smarr relation is then reached by considering the dimension of each of the quantities. It is an application of Euler's theorem on quasi homogeneous functions. The mass goes as L, the entropy as L^2, P as L^{-2}, and q as L. We would then get the following relationship.

$$M = 2TS - 2VP + q\phi \qquad (3.45)$$

It is easy to check that the above relationship holds.

Chapter 4

Taub-NUT Spacetimes

In this chapter we will be introducing the Taub-NUT metric in the most simple case possible. We will then explore its richness and how it varies when we consider a non-zero cosmological constant, as well as a charged case. Mainly we will examine how the charges and the horizon change for each spacetime.

It's important to note that for calculations for charges in this chapter and after, metrics that do not have a spherical boundary will pick up some coefficient when integrating over the non-r coordinates. This term will be set to unity since it is dependent on the shape of the non-r related boundary and can always be normalised by dividing by its area. The motivation behind this is to compare between the family of spacetimes, differentiated by the metric the value of k.

We will first go over the spherical Taub-NUT metric in its most prevalent form in the literature. Afterwards, we will introduce a k parameter dependent solution. The value of k will describe the horizon geometry, and the asymptotic behaviour of the metric. We will look at the flat, hyperbolic, and spherical spacetimes.

4.1 Spherical Metric

The metric was first discovered in 1951[17] and is a vacuum axisymmetric exact solution to Einstein's field equations. However, it was expressed in coordinates that only described the time-dependent portion of the spacetime. Newman, Unti, and Tamburino, extended the solution in 1963 to the metric we know of today.[18] Their initials represent the NUT portion in the name of the solution.[19] In this chapter we will be discussing the metric as it exists within the literature.

The defining property of the metric is the nut parameter n. It is the reason that the metric is not asymptotically flat. It is interesting to note that it was one of the parameters found by Roy Kerr in his attempt to find a rotating, and axisymmetric solution. However, he dismissed it as a parameter that was not physical.[20] This was because the boundary metric does not match that of Minkowski spacetime due to the persistence of the nut parameter in the boundary metric.

The spacetime is quite rich and has been the cause for much debate. We will begin by considering the metric with it angular components corresponding to that of a 2-sphere. In this case, upon taking the limit $\lim_{n \to 0}$ we revert to the Schwarzschild metric. We will be progressively considering more complications to the spacetime. The metric in this form is expressed as

$$ds^2 = -f(r)\Big(dt - 2n\, cos\, \theta d\phi\Big)^2 + \frac{1}{f(r)} dr^2 + (r^2 + n^2)\Big(d\theta^2 + sin^2\theta d\phi^2\Big) \quad (4.1)$$

With

$$f(r) = \frac{r^2 - 2mr - n^2}{r^2 + n^2} \quad (4.2)$$

For the sake of quick analysis, let us revert to the Cartesian coordinate system. Along the z axis, the components of the $dtd\phi$ portion and the $d\phi^2$ do not vanish at the axes which causes the main problem here as the ϕ coordinate is degenerate along the axis. This corresponds to the $\theta = 0$ and the $\theta = \pi$ lines.

Since ϕ is degenerate along the z axis then any coefficient for $d\phi$ should vanish there. As the z axis above and below are seperated by the black hole itself the singular behaviour occurs along 2 distinct ”strings”. We cannot regularise the behaviour along both sides of the z axis at the same time.

Calling them strings aims to show that they are analogous to the Dirac string. Dirac had tried to find a cause for the quantisation of the electric charge. This lead to him hypothesising the existence of a magnetic monopole. If it existed, electrical charge must be quantized.

This magnetic charge, however, results in the existence of a Dirac ”string” along the z axis in one of the hemispheres around the magnetic charge. This comes from choosing the vector potential to be defined along the northern or southern hemisphere surrounding the charge. Whenever we choose the vector potential along one, the other will exist. The solution to this is a phase condition that makes the string invisible. What Misner tried to do for the strings in the Taub-NUT spacetime was analogous.

Misner was the first person to note that we could cover the whole spacetime outside the horizon.[21] The way to do this was to take two different patches each with its own time coordinate. We would shift the time coordinate for both of them. With the time for the positive z axis denoted t^+, and the negative one t^-. Misner proposed the following transformation for the patches. [22]

$$\begin{aligned} t &= t^+ + 2n\phi \\ t &= t^- - 2n\phi \end{aligned} \tag{4.3}$$

To keep our spacetime compact, both patches need to equate to each other at the equator. To do this, we simply set both times on the left hand side equal to each other. This results in

$$t^- = t^+ + 4n\phi \tag{4.4}$$

This introduces a serious problem for our time coordinate. Let us consider the consequences of the equation above. The ϕ coordinate, being an angular one, is periodic with period 2π. This results in a periodicity of the time coordinate. If we take a 2π path along ϕ we would be back to the same point in time. This then results in a periodicity of $8\pi n$ for t.

The adoption of the above patches would completely hide the strings. This is how they came to be known as Misner strings. As highlighted above this periodicity condition will introduce closed timelike curves everywhere. This would be a serious problem from a physical and causal sense.

Due to concerns emanating from the Misner strings, the metric was usually quoted in the euclidean form.[23–29] This treatment fully embraces the time periodisation condition. With the euclidean Wick rotation, the terms including a squared nut charge also change their signs. We, then, have to limit ourselves to the portion where $r_+ > n$ as the angular portion of the metric will flip its sign for this range of r_+. The Wick rotation takes the following form

$$t \to i\tau$$
$$n \to i\bar{n} \tag{4.5}$$

For the euclidean case, the solutions are usually segregated into two distinct solutions. One for $r_h = n$, which is the lower limit for r_h. These are called nut solutions. Solutions where $r_h > n$ are called bolt ones.[29–31] The nuts are commonly used as reference spacetimes for subtraction from the bolt solutions. In the Lorentzian case, we do not have any problem with $r_h < n$.

The problem that is introduced with the hiding of the Misner strings is that a big reason to disregard them was the possibility of closed time loops and causality violation along the z axis. It was recently proven [32] that the strings are transparent to geodesics passing through them.

Furthermore, they proved that for a family of parameter-dependent solutions no closed timelike or null geodesics are able to violate causality. Thus, the ”strings” themselves don’t necessarily violate causality while hiding them does.

One aspect of the time coordinate periodicity is how it affects the thermodynamics of the solution. The periodicity constraint restrains the temperature. This decreases our degrees of freedom, and limits the phase structure.

It’s important to note that in some treatments, including ours, the nut parameter n is taken to represent a gravito-magnetic charge that is analogous to the magnetic charge. This is justified through the calculation of the electric and magnetic charges using Komar integrals. A similar correspondence between the mass and the nut charge exists.

The mass of a stationary spacetime can be calculated through the use of forms. Forms and Komar integrals are tackled in appendix A. The mass can be calculated by integrating over the hodge dual of the exterior derivative of the one-form generated by the timelike killing vector ξ.

$$-\frac{1}{4\pi}\int_{S^2_\infty} \star d\xi = m \tag{4.6}$$

The reason the nut parameter is interpreted as a gravitomagnetic mass is that we can find the nut charge of a spacetime through integrating over the 2-form $d\xi$. Which is the hodge dual to the form that was used in the mass calculation.

$$\frac{1}{4\pi}\int_{S^2_\infty} d\xi = -n \tag{4.7}$$

Let us consider the event horizon. We will do so by looking at $f(r) = 0$. This will show that we have two separate horizons, a larger and a smaller one. Outside the larger radial horizon r_+ the r coordinate is spacelike, between r_+, and the smaller r_- it is timelike. For $r < r_-$ it reverts to being spacelike.

$$r_\pm = m \pm \sqrt{m^2 + n^2} \tag{4.8}$$

4.2 A More General Solution

A more general metric for the Taub-NUT spacetime can be reached by solving the field equations. The solution depends on the parameter k and it defines the shape of the horizon. We assumed the following

$$ds^2 = -f(r)\Big(dt + 2ng(x)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2+n^2)\Big(\frac{dx^2}{1-kx^2} + x^2 d\phi^2\Big) \tag{4.9}$$

Our solution will be that of a vacuum one. The field equations take the form

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu} \\ &= 0 \end{aligned} \tag{4.10}$$

Upon solving the equations we find the form for $g(x)$ and $f(r)$.

$$\begin{aligned} g(x) &= \frac{-2n\sqrt{1-kx^2}+c}{2kn} \\ f(r) &= \frac{k(r^2-n^2)-2mr}{r^2+n^2} \end{aligned} \tag{4.11}$$

There are three particular values of k that give distinct horizon geometries. We will show that each of these corresponds to a particular horizon geometry. We will consider the values $k = -1, 0, 1$. The effect on $f(r)$ is obvious as the value of k will dictate the role of the $r^2 - n^2$ term.

The addition of k-dependence to the metric allows us to examine the difference between the horizons in all three geometries. Below we can see the dependence of the horizon radii on k. It is important to note that without a cosmological constant we are not interested in the flat case. This is because the r coordinate is timelike everywhere.

$$r_{\pm} = m \pm \sqrt{m^2 + k^2 n^2} \tag{4.12}$$

For the $k = 1$ and the $k = -1$ cases, the horizon does not differ in its position in terms of the radial coordinate r.

Let us consider the effect on $g(x)$. In the cases where $k = 1$ and $k = -1$ we only need to take x through a coordinate transformation. In the case where $k = 1$ we will take $x = sin(\theta)$ and $x = sinh(\theta)$ for the value of $k = -1$. We then get the following forms for $g(x)$

$$\begin{aligned} g_{\text{k=-1}} &= 2n(cosh(\theta) + c) \\ g_{\text{k=1}} &= -2n(cos(\theta) + c) \end{aligned} \tag{4.13}$$

For $k = -1$ and $k = 1$ we will show the necessary coordinate transformations

$$\begin{aligned} x &\rightarrow cos(\theta) \\ k &\rightarrow 1 \\ \sqrt{1 - kx^2} &= \sqrt{1 - cos^2(\theta)} \\ \sqrt{1 - cos^2(\theta)} &= sin(\theta) \\ dx &= -sin(\theta) d\theta \end{aligned} \tag{4.14}$$

While for the case of $k = -1$

$$\begin{aligned} x &\rightarrow cosH(\theta) \\ k &\rightarrow 1 \\ \sqrt{1 - kx^2} &= \sqrt{1 + cosh^2(\theta)} \\ \sqrt{1 + cosh^2(\theta)} &= sinh(\theta) \\ dx &= sinh(\theta) d\theta \end{aligned} \tag{4.15}$$

We arrive at the metrics[19]

$$ds^2_{\text{k=-1}} = -f(r)\Big(dt + 2n(cosh(\theta) + c)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\Big(d\theta^2 + sinh^2(\theta)d\phi^2\Big)$$
$$ds^2_{\text{k=1}} = -f(r)\Big(dt - 2n(cos(\theta) + c)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\Big(d\theta^2 + sin^2(\theta)d\phi^2\Big) \tag{4.16}$$

It is important to note that a careful choice of constants could make the behaviour regular along the line where ϕ is degenerate for the case of $k = -1$. However, this is not possible for $k = 1$. We inevitably need two patches if we want to hide the misner strings as we said above.

4.3 Dyonic Taub-NUT

For the dyonic case we will be examining the effect of adding electric and magnetic charges to our formulation of the metric. The addition of the electromagnetic charges will affect our solution through changing the field equations and hence imposing some additional restrictions on our solution.

The form of $g(x)$ does not change but there are different restrictions on the value c could take. The value of $f(r)$, however, changes explicitly. We can express it as

$$f(r) = \frac{k(r^2 - n^2) - 2mr + q^2 + p^2}{r^2 + n^2} \tag{4.17}$$

The horizon radius then changes to

$$r_{\pm} = m \pm \sqrt{m^2 + k^2n^2 - k(q^2 + p^2)} \tag{4.18}$$

This then introduces relative values for our parameters where r does not exist. Whenever the square root is not real, there is no root for $f(r)$. Similar to the sner-Nordström case, there will be an extremal solution that occurs when

$$m^2 + k^2n^2 = k(q^2 + p^2) \tag{4.19}$$

It's important to note, the case when $k = 0$ here, results in an event horizon. However, it cannot be given by the above equation since it assumes a quadratic nature and is not defined when the leading coefficient is zero. This horizon, however, isn't generally considered as such since the spacetime as a whole has a timelike r coordinate everywhere in the limit of dying electromagnetic charges. However, we will include the $k = 0$ case as the analysis would hold for the AdS case where we consider it.

We will now consider the effects of the electromagnetic charges on the metric and their relation to each other. We solve for the one-form A. Along with the restrictions on the metric we get some restrictions on A that are forced by the thermodynamics.

$$\begin{aligned} A_t &= \frac{(np + n^2V - qr + r^2V)}{(r^2 + n^2)} \\ A_\phi &= \frac{p\sqrt{1 - kx^2} + d \cdot k}{k} + \frac{(np - qr)\left(\frac{-2n\sqrt{1-kx^2}}{k} + c\right)}{(r^2 + n^2)} \end{aligned} \tag{4.20}$$

These restrictions have to do with fixing the boundary conditions for the path integral used in finding the temperature. The A_t component must vanish at the horizon. The A_ϕ component needs to vanish along the line where ϕ is degenerate. This results in different restrictions on the constants for each value of k.

Calculating the contraction of the one-form A with itself $A_\mu A^\mu$, allows us to see the solution within one equation. The restriction imposed by the A_t portion will be the same for all spacetimes, while the restriction imposed by the A_ϕ component will differ.

The shared restriction couples the value of the electric charge at infinity to the magnetic charge p, nut charge n, and the electric potential V. The relationship is given by

$$q = \frac{pn + V(r^2 + n^2)}{r} \tag{4.21}$$

For the A_ϕ to not be singular for the case of $k = 0$, we will need to assign a particular value to d such that the fraction does not tend to infinity in the limit where k approaches zero. As such, for $k = 0$ it will need to take the following form

$$d = \frac{-p}{k} + d' \tag{4.22}$$

For each value of k, A_ϕ takes the explicit form

$$
\begin{aligned}
[A_\phi]_{\text{k=1}} &= p \cdot cos\,\theta + d + \frac{(np - qr)\big(- 2n \cdot cos\,\theta + c\big)}{(r^2 + n^2)} \\
[A_\phi]_{\text{k=0}} &= -p \cdot \frac{x^2}{2} + d' + \frac{(np - qr)\big(nx^2 + c'\big)}{(r^2 + n^2)} \\
[A_\phi]_{\text{k=-1}} &= p \cdot cosh\,\theta + d + \frac{(np - qr)\big(2n \cdot cosh\,\theta + c\big)}{(r^2 + n^2)}
\end{aligned}
\tag{4.23}
$$

It is very important to note that the constant represented by c is the very same one that shows up in $g(x)$. Thus, our choice of constants could affect the metric. This relationship between c and d then dictates the amount of patches we need to use to cover the spacetime. The restrictions imposed results in

$$
\begin{aligned}
c_{\text{ k=1}} &= \frac{d_{\text{ k=1}} \pm (p + 2nV)}{V} \\
c'_{\text{ k=0}} &= \frac{d'_{\text{ k=0}}}{V} \\
c_{\text{ k=-1}} &= \frac{d_{\text{ k=-1}} - (p + 2nV)}{V}
\end{aligned}
\tag{4.24}
$$

As is seen above, two patches are necessary to describe the spherical horizon. This is represented in the $\pm$ sign in the relation. The sign is positive for the case where $\theta = 0$ and is negative when $\theta = \pi$. This will lead us to use two patches, particularly in the calculation of charges.

In contrast, the cases where $k = 0$ and $k = -1$ only require a careful choice of constants. Since the assumptions used to solve the field equations only assumed that A_ϕ is a function in r and x, any other parameter would not violate the field equation.

Our choice should let the solution revert to the sner-Nordström case when n goes to zero. Another subtlety, that is equally important, is to make sure that our choice is sound dimensionally. The choice of constants should have the same dimension as the rest of the equations they are part of.

Let us recall the analogy we drew between the electromagnetic charges, the mass and the nut charge. This owes to how we calculate the electric and magnetic charges. Let us take the one-form A_μ presented above, operating on it with the exterior derivative we get

$$dA = F \tag{4.25}$$

Taking the hodge dual of the two-form F

$$\star F = H \tag{4.26}$$

To calculate the electric charge within the spacetime we use

$$\frac{1}{4\pi}\int_{S^2_\infty} \star F = q \tag{4.27}$$

While

$$\frac{-1}{4\pi}\int_{S^2_\infty} F = p \tag{4.28}$$

This is exactly what happened for the gravitomagnetic case above. Integrating one two-form over the sphere at infinity gives us one of the charges while integrating the other gives us its dual charge.

4.4 Taub-NUT-AdS

In this sub-chapter, we will discuss the addition of a cosmological constant Λ to our spacetime and changes that follow from that. We will examine how the horizons differ for each geometry. For the AdS case, the field equations from the uncharged case are supplemented with a cosmological constant.

$$T_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu}\Big(\Lambda - \frac{1}{2}R\Big) \tag{4.29}$$

This metric is solved by adding a cosmological constant to the field equations and solving them. The solution is longer a vacuum one and it gets a contribution from the cosmological constant in the function $f(r)$.

The metric takes the same form as it did above, the only portion of the metric that will change is $f(r)$. We present $f(r)$ below

$$f(r) = \frac{(r^2 - n^2) - 2mr - \Lambda(\frac{1}{3}r^4 + 2n^2r^2 - n^4)}{(r^2 + n^2)} \tag{4.30}$$

Taking

$$\Lambda = \frac{-3}{l^2} \tag{4.31}$$

We get

$$f(r) = \frac{l^2(-2mr + k(r^2 - n^2)) + r^4 + 6n^2r^2 - 3n^4}{l^2(r^2 + n^2)} \tag{4.32}$$

Quartic polynomials, are the highest degree of polynomials we can regularly solve. However, they do not make for concise or comprehensible solutions by any means.

It is convention that when we work with a polynomial of a high degree where an analytic solution doesn't exist that we solve for the mass m that corresponds to the horizon at $f(r) = 0$. Following this method, the mass would be equal to

$$m = \frac{l^2\big(k \cdot (r_h^2 - n^2)\big) + r_h^4 + 6n^2r_h^2 - 3n^4}{2r_hl^2} \tag{4.33}$$

As there is only one horizon we are considering, henceforth we will be denoting all horizon radii with r_h as done above. There was no implicit assumption on k in our solution this time, thus the above form holds for all values of k. It can be seen from the equations above, we have event horizons for the three values k can take.

The existence of the cosmological constant does not affect the constant in $g(x)$. Seeing as it does not interfere with any of the $d\phi$ components of the metric, this is expected. Thus, it does not enforce any conditions that could alter the number of patches needed to fully describe a spacetime outside the horizon.

4.5 Dyonic Taub-NUT-AdS

We end this chapter with a combination of the properties of the electromagnetic Taub-NUT spacetime and the Taub-NUT-AdS spacetime. As we saw above, neither particularly changed the properties of the other, so the transition will be fairly smooth.

The metric will now retain both the electromagnetic and cosmological constant contributions. The field equations are now quite different from the vacuum solution.

The coupled equations can be represented by

$$\begin{aligned} T_{\mu\nu} &= F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \\ T_{\mu\nu} &= R_{\mu\nu} + g_{\mu\nu}\Big(\Lambda - \frac{1}{2}R\Big) \end{aligned} \tag{4.34}$$

The forms for A_μ are not different from the ones above. We end up seeing the same relations between both the constants in the metric and in A_ϕ. The coupling between q and p through n retains its form.

The main difference is the contributions from both the electromagnetic charges and cosmological constant that are added to $f(r)$. This then changes the mass at which $f(r) = 0$, and hence r_h itself. $f(r)$ takes the form

$$f(r) = \frac{l^2(p^2 + q^2 - 2mr + k(r^2 - n^2)) + r^4 + 6n^2r^2 - 3n^4}{l^2(r^2 + n^2)} \tag{4.35}$$

Resulting in an asymptotic mass

$$m = \frac{l^2\big(p^2 + q^2 + k \cdot (r_h^2 - n^2)\big) + r_h^4 + 6n^2r_h^2 - 3n^4}{2r_h l^2} \tag{4.36}$$

Just to make sure that the addition of the cosmological constant has not affected the electromagnetic charges we calculate the charges to find

$$\begin{aligned} \frac{1}{4\pi}\int_{\partial\Sigma} \star F &= q \\ \frac{-1}{4\pi}\int_{\partial\Sigma} F &= p \end{aligned} \tag{4.37}$$

Let us then make sure that our analogy holds for n as a gravitomagnetic mass. As we did for the initial Taub-NUT solution, we integrate the two-forms $d\xi$ and $\star d\xi$ over the boundary at infinity.

$$\begin{aligned} \frac{-1}{4\pi}\int_{\partial\Sigma} \star d\xi &= \infty \\ \frac{1}{4\pi}\int_{\partial\Sigma} d\xi &= -\infty \end{aligned} \tag{4.38}$$

They both diverge! This happens due to the factor that goes as r^2 that comes with the cosmological constant. To solve this we use the method introduced in [33] to tackle the problem. This method was used in the spherical case, albeit in a different formulation to ours in [34, 35] It was also used in [36] for the spherical case.

This is a result of adding the cosmological constant. Without it, $f(r)$ goes as $\frac{r^2}{r^2}$ for large r. However when we add a cosmological constant, we find that $f(r)$ starts behaving as $\frac{r^4}{r^2}$ which causes the divergence.

The solution to this is the use of another conserved two-form to cancel the divergences. We do so by considering the two form ω. It is related to the timelike killing vector ξ through

$$\begin{aligned} d\omega &= \star\, \xi \\ (d\omega)_{\mu\nu\rho} &= \epsilon_{\lambda\mu\nu\rho} g^{\delta\lambda} \xi_{\delta} \end{aligned} \tag{4.39}$$

The mass and gravitomagnetic charges can then be calculated, we do it for the spherical case here

$$\begin{aligned} \frac{-1}{4\pi} \int_{S^2_\infty} d\xi + 2\Lambda \star \omega &= m \\ \frac{1}{4\pi} \int_{S^2_\infty} d\xi - 2\Lambda \star \omega &= -n\left(1 + \frac{4n^2}{l^2}\right) \end{aligned} \tag{4.40}$$

The hodge dual of the mass is no longer just the nut parameter n. We can define a new charge N that represents the gravitomagnetic charge above. For now it is interesting to note that this charge will differ across the three spacetimes. This will be discussed at more length in chapter 5.

Chapter 5

Taub-NUT AdS Thermodynamics

In chapter 4 we gave our attention to the Taub-NUT solution and especially to a family of solutions that were parameterised through a dependence on k. This dependence changes the shape of the horizon. In this chapter we will see the effects of this parameter on the thermodynamics.

We will first go through the dependence on k, highlight the differences in the metrics and some of the issues with dealing with the flat metric. We will also investigate how the constants in the metric and in A_ϕ come into play.

5.1 A Deeper Look

In this sub-chapter we will recall what we found about the k-dependence of the metric. Some of the results we presented in chapter 4 will be explained and derived in more detail. We will delve deeper into the details of how these set the spacetimes apart. Below, we see the most generic form of the k-dependent metric.

$$ds^2 = -f(r)\Big(dt + 2ng(x)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\Big(\frac{dx^2}{1 - kx^2} + x^2 d\phi^2\Big) \quad (5.1)$$

Under the transformations over x and defining the value for k we have shown that we reach

$$
\begin{aligned}
ds^2_{\text{k=-1}} &= -f(r)\Big(dt + 2n(cosh(\theta) + c)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\Big(d\theta^2 + sinh^2(\theta)d\phi^2\Big) \\
ds^2_{\text{k=0}} &= -f(r)\Big(dt + 2n(x^2 + c)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\Big(dx^2 + x^2 d\phi^2\Big) \\
ds^2_{\text{k=1}} &= -f(r)\Big(dt - 2n(cos(\theta) + c)d\phi\Big)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\Big(d\theta^2 + sin^2(\theta)d\phi^2\Big)
\end{aligned} \tag{5.2}
$$

Where

$$
f(r) = \frac{l^2(p^2 + q^2 - 2mr + k(r^2 - n^2)) + r^4 + 6n^2r^2 - 3n^4}{l^2(r^2 + n^2)} \tag{5.3}
$$

A universal constraint on all of the solutions is the one imposed by requiring that $A_\mu A^\mu$ vanishes at the horizon. This results in the relation

$$
q = \frac{pn + \Phi_e(r_h^2 + n^2)}{r_h} \tag{5.4}
$$

It is interesting that the above coupling does not depend on the value of k. As we argued above, a careful choice of constants in the relation between A_ϕ and the generic $g(x)$ could result in a spacetime where the whole region outside the black hole horizon can be described using one patch. This holds in the flat and hyperbolic cases but does not hold for the spherical metric where only one of the strings is hideable at a time. Thus, at least two patches are needed to fully cover the spacetime. Let us recall the relationship between the constants

$$
\begin{aligned}
c_{\text{ k=1}} &= \frac{d_{\text{ k=1}} \pm (p + 2n\Phi_e)}{\Phi_e} \\
c_{\text{ k=0}} &= \frac{d_{\text{ k=0}}}{\Phi_e} \\
c_{\text{ k=-1}} &= \frac{d_{\text{ k=-1}} - (p + 2n\Phi_e)}{\Phi_e}
\end{aligned} \tag{5.5}
$$

5.2 Electromagnetic Charges and Potentials

As stated above, we start by fixing the one-form A at the boundaries. This will reflect in our thermodynamics later on as we will be fixing the electric potential and magnetic charge at the boundaries.

We will start from A, then operate on it with the exterior derivative to get dA. We then calculate the hodge dual of dA. Taking the hodge dual of dA to be dB, we then calculate the one-form B. We use these four forms to calculate the electromagnetic charges and potentials. Writing A in form notation

$$
\begin{aligned}
A_{\text{k=1}} &= \frac{np - qr + \Phi_e(r^2+n^2)}{r^2+n^2}dt + \left(p \cdot cos\,\theta + d + \frac{(np-qr)\big(-2n \cdot cos\theta + c\big)}{(r^2+n^2)}\right)d\phi \\
A_{\text{k=0}} &= \frac{np - qr + \Phi_e(r^2+n^2)}{r^2+n^2}dt + \left(p \cdot \frac{-x^2}{2} + d' + \frac{(np-qr)\big(nx^2 + c'\big)}{(r^2+n^2)}\right)d\phi \\
A_{\text{k=-1}} &= \frac{np - qr + \Phi_e(r^2+n^2)}{r^2+n^2}dt + \left(p \cdot cosh\theta + d + \frac{(np-qr)\big(2n \cdot cosh\,\theta + c\big)}{(r^2+n^2)}\right)d\phi
\end{aligned}
\tag{5.6}
$$

We then compute dA using the method explained in appendix A.

$$
\begin{aligned}
dA_{k=-1} =& \frac{2(n^2q + 2npr - qr^2)}{(n^2+r^2)^2}dt \wedge dr \\
& \frac{2(2ncosh\theta + c')(n^2q + 2npr - qr^2)}{(n^2+r^2)^2}dr \wedge d\phi \\
& \frac{2(2ncosh\theta + c1)(n^2p - 2nqr - pr^2)}{(n^2+r^2)^2}d\theta \wedge d\phi \\
dA_{k=0} =& \frac{2(n^2q + 2npr - qr^2)}{(n^2+r^2)^2}dt \wedge dr \\
& \frac{(2nx^2 + c')(n^2q + 2npr - qr^2)}{(n^2+r^2)^2}dr \wedge d\phi \\
& \frac{2x(n^2p - 2nqr - pr^2)}{(n^2+r^2)}d\theta \wedge d\phi \\
dA_{k=1} =& \frac{2(n^2q + 2npr - qr^2)}{(n^2+r^2)^2}dt \wedge dr \\
& \frac{2(2ncos\theta + c')(n^2q + 2npr - qr^2)}{(n^2+r^2)^2}dr \wedge d\phi \\
& \frac{2cos\theta(n^2p - 2nqr - pr^2)}{(n^2+r^2)}d\theta \wedge d\phi
\end{aligned}
\tag{5.7}
$$

We then get the hodge duals $\star dA$

$$\begin{aligned}\star dA_{k=-1} =&2\frac{n^2p-2nqr-pr^2}{(n^2+r^2)^2}dt\wedge dr\\&\frac{-2(2ncosh\theta+c')(n^2p-2nqr-pr^2)}{(n^2+r^2)^2}dr\wedge d\phi\\&\frac{-2sinh\theta(n^2q+2npr-qr^2)}{(n^2+r^2)}d\theta\wedge d\phi\\\star dA_{k=0} =&2\frac{n^2p-2nqr-pr^2}{(n^2+r^2)^2}dt\wedge dr\\&\frac{-2(nx^2+c')(n^2p+2nqr-pr^2)}{(n^2+r^2)^2}dr\wedge d\phi\\&\frac{2x(qr^2-n^2q-2npr)}{(n^2+r^2)}d\theta\wedge d\phi\end{aligned} \tag{5.8}$$

$$\begin{aligned}\star dA_{k=1} =&2\frac{n^2p-2nqr-pr^2}{(n^2+r^2)^2}dt\wedge dr\\&\frac{2(2ncos\theta+c')(n^2p-2nqr-pr^2)}{(n^2+r^2)^2}dr\wedge d\phi\\&\frac{-2sin\theta(n^2q+2npr-qr^2)}{(n^2+r^2)}d\theta\wedge d\phi\end{aligned} \tag{5.9}$$

Let us rename the two-form $\star dA$ to dB. We then need to solve the following tensor equation to get B.

$$\left(dB\right)_{\mu\nu}=\left(\partial_\mu B_\nu-\partial_\nu B_\mu\right) \tag{5.10}$$

This then results in a one form B

$$\begin{aligned}B_{\text{k=1}} &= 2\frac{-np-qr+c_{m_1}(r^2+n^2)}{r^2+n^2}dt+2\left(pcos\,\theta+c_{m_1}+\frac{(nq+pr)\big(-2n\cdot cos\theta+c\big)}{(r^2+n^2)}\right)d\phi\\B_{\text{k=0}} &= \frac{-np-qr+c_{m_1}(r^2+n^2)}{r^2+n^2}dt+\left(p\frac{-x^2}{2}+c_{m_2}+\frac{(nq+pr)\big(nx^2+c'\big)}{(r^2+n^2)}\right)d\phi\\B_{\text{k=-1}} &= \frac{-np-qr+c_{m_1}(r^2+n^2)}{r^2+n^2}dt+\left(pcosh\theta+c_{m_2}+\frac{(nq+pr)\big(2n\cdot cosh\,\theta+c\big)}{(r^2+n^2)}\right)d\phi\end{aligned} \tag{5.11}$$

We are now in a position where we can calculate the charges at a constant-r surface. Over each boundary there will be a term that has to do with the area of the space we are integrating over. We will be setting this factor to unity as it carries little significance in terms of comparing between the properties of the three spacetimes.

$$
\begin{aligned}
q(r) &= \int_{\partial\Sigma} dB = \frac{q^2(r^2 - n^2 - 2npr)}{n^2 + r^2} \\
p(r) &= \int_{\partial\Sigma} dA = \frac{p^2(r^2 - n^2 + 2nqr)}{n^2 + r^2}
\end{aligned}
\tag{5.12}
$$

The change of each charge as r change indicates that there is some charge distribution between the horizon and the boundary at arbitrarily large r. This was explained for the spherical case [37] by the existence of the charges along the misner strings, however there are no strings for the flat horizon in particular. This hints that the main culprit is the nut charge. There is a coupling between the charges that shows when we compare the charges at infinity with the charges on the horizon.

$$
\begin{aligned}
q_h &= \frac{\Phi_e(r_h^2 - n^2) - np}{r_h} \\
p_h &= (p + 2n\Phi_e)
\end{aligned}
\tag{5.13}
$$

The picture will become a bit clearer in a moment, but we will need to calculate the electromagnetic potentials first. To calculate the potential we contract the corresponding one-form with the timelike killing vector and check the potential difference between both boundaries. It is important to note that we cannot define B uniquely. However, since we are only interested in the difference between the horizon and infinity this is not a problem. This gives us

$$
\begin{aligned}
\xi^\mu A_\mu\Big|_{r\to\infty} - \xi^\mu A_\mu\Big|_{r=r_h} &= \Phi_e \\
\xi^\mu B_\mu\Big|_{r\to\infty} - \xi^\mu B_\mu\Big|_{r=r_h} &= \Phi_m = \frac{p + nV}{r_h}
\end{aligned}
\tag{5.14}
$$

It is easy to verify that the charges at the horizon and at infinity are related by

$$
\begin{aligned}
q_h &= q - 2n\Phi_m \\
p_h &= p + 2n\Phi_e
\end{aligned}
\tag{5.15}
$$

5.3 Gravito-Magnetic Charges

We will be calculating the mass and the nut charge. Starting from the killing vectorξ^μ, we will write it explicitly as a one-form ξ before taking its exterior derivative. After finding $d\xi$ we will then find its hodge dual. We then find ω such that the integrals for the mass and nut charge are finite.

$$\begin{aligned}\xi_{k=-1} &= -f(r)dt - f(r)(2ncosh\theta + c)d\phi \\ \xi_{k=0} &= -f(r)dt - f(r)(2nx^2 + c)d\phi \\ \xi_{k=1} &= -f(r)dt + f(r)(2ncos\theta + c)d\phi\end{aligned} \tag{5.16}$$

Applying the exterior derivative gives us

$$\begin{aligned}d\xi_{k=-1} &= 2f'(r)(dt \wedge dr) - 2f'(r)(2ncosh\theta + c)(dr \wedge d\phi) - 4nf(r)sinh\theta(d\theta \wedge d\phi) \\ d\xi_{k=0} &= 2f'(r)(dt \wedge dr) - 2f'(r)(2nx^2 + c')(dr \wedge d\phi) - 4nf(r)x(dx \wedge d\phi) \\ d\xi_{k=1} &= 2f'(r)(dt \wedge dr) + 2f'(r)(2ncos\theta + c)(dr \wedge d\phi) - 4nf(r)sin\theta(d\theta \wedge d\phi)\end{aligned} \tag{5.17}$$

Taking the hodge dual

$$\begin{aligned}\star d\xi_{k=-1} = &\frac{-4nf(r)}{r^2+n^2}(dt \wedge dr) + \frac{4f(r)(ncosh\theta + c')n}{n^2+r^2})(dr \wedge d\phi) \\ &- 2nf'(r)sinh\theta(n^2+r^2)(d\theta \wedge d\phi) \\ \star d\xi_{k=0} = &\frac{-4nf(r)}{r^2+n^2}(dt \wedge dr) + \frac{4f(r)(nx^2 + c')n}{n^2+r^2}(dr \wedge d\phi) \\ &- 2nf'(r)x(n^2+r^2)(dx \wedge d\phi) \\ \star d\xi_{k=1} = &\frac{-4nf(r)}{r^2+n^2}(dt \wedge dr) - \frac{4f(r)(ncos\theta + c')n}{n^2+r^2})(dr \wedge d\phi) \\ &- 2nf'(r)sin\theta(n^2+r^2)(d\theta \wedge d\phi)\end{aligned} \tag{5.18}$$

We now want to find ω so that the integrals for the mass and the nut charge do not diverge. We first calculate $d\omega$ through

$$\begin{aligned}d\omega &= \star\,\xi \\ (d\omega)_{\mu\nu\rho} &= \epsilon_{\lambda\mu\nu\rho} g^{\delta\lambda} \xi_\delta\end{aligned} \tag{5.19}$$

To calculate ω we need to solve the differential equations that result from

$$(d\omega)_{\mu\nu\rho} = 3\nabla_{[\mu}\omega_{\nu\rho]} \tag{5.20}$$

Though, ω is not uniquely defined, we use this to make sure that our form for it cancels the divergences for both the mass and the nut charge. The two-form for each value of k take the form

$$\begin{aligned}
\omega_{k=-1} &= \frac{2n}{3}(dt\wedge dr) - \frac{4n^2}{3}cosh\theta(dr\wedge d\phi) + \frac{2r}{3}(r^2+n^2)(d\theta\wedge d\phi) \\
\omega_{k=0} &= \frac{2n}{3}(dt\wedge dr) - \frac{2x^2n^2}{3}(dr\wedge dx) + \frac{2xr}{3}(r^2+n^2)(dx\wedge d\phi) \\
\omega_{k=1} &= \frac{2n}{3}(dt\wedge dr) + \frac{4n^2}{3}cos\theta(dr\wedge d\phi) + \frac{2r}{3}(r^2+n^2)(d\theta\wedge d\phi)
\end{aligned} \tag{5.21}$$

The last form we need to consider before calculating the mass and the nut charge is the dual to ω. All that remains is to operate on ω with the hodge operator.

$$\begin{aligned}
\star\omega_{k=-1} &= \frac{2r}{3}(dt\wedge dr) - \frac{2cf(r)n}{3sinh\theta}(dt\wedge d\phi) - \frac{2r(2ncosh\theta+c)}{3}(dr\wedge d\phi) \\
&- \frac{2n\Big((n^2+r^2)cosh^2\theta - 2cnf(r)cosh\theta - f(r)c^2 - n^2 - r^2\Big)}{3sinh\theta}(d\theta\wedge d\phi) \\
\star\omega_{k=0} &= \frac{2r}{3}(dt\wedge dr) - \frac{2cf(r)n}{3x}(dt\wedge d\phi) - \frac{2r(2nx^2+c')}{3}(dr\wedge d\phi) \\
&+ \frac{2n\Big(c'(nx^2+c')f(r) - (n^2+r^2)x^2\Big)}{3x}(dx\wedge d\phi) \\
\star\omega_{k=1} &= \frac{2r}{3}(dt\wedge dr) + \frac{2cf(r)n}{3sin\theta}(dt\wedge d\phi) + \frac{r(4ncos\theta+c)}{3}(dr\wedge d\phi) \\
&+ \frac{2n\Big((n^2+r^2)cos^2\theta + c2nf(r)cos\theta + f(r)c^2 - n^2 - r^2\Big)}{3sin\theta}(d\theta\wedge d\phi)
\end{aligned} \tag{5.22}$$

All that remains now is to carry out the integral of the 2-forms over the boundary. It is important to note that one of the conditions for convergence is the vanishing of c. This is the first time we find a physical charge that depends on c. The mass for all three spacetimes is the same. but the gravitomagnetic charge differs.

$$\frac{-1}{4\pi}\int_{\partial M}\star d\xi + 2\Lambda\omega = m \tag{5.23}$$

The gravitomagnetic charge N takes the following form

$$\begin{aligned} N_{k=-1} =& \frac{1}{4\pi}\int_{\partial M} d\xi - 2\Lambda \star \omega = n\left(1 - \frac{4n^2}{l^2}\right) \\ N_{k=0} =& \frac{1}{4\pi}\int_{\partial M} d\xi - 2\Lambda \star \omega = -\frac{n^3}{l^2} \\ N_{k=1} =& \frac{1}{4\pi}\int_{\partial M} d\xi - 2\Lambda \star \omega = -n\left(1 + \frac{4n^2}{l^2}\right) \end{aligned} \tag{5.24}$$

One important quantity that we have not discussed, yet, is the angular momentum. As with the conserved quantities above, we calculate it through a two-form on the boundary. To get that two-form we first take the exterior derivative of the killing vector $\chi^\mu = \partial_\phi$.

As was noted in [38, 39]the angular momentum depends on the value of the constant c. Above we noted that the condition for the non-divergence of the nut charge N depends on the vanishing of this constant. This is also the case for the angular momentum. It was noted that if we subtract an $m = 0$ solution we get a finite value of $3cmn$.

We will not be treating the angular momentum as one of our thermodynamic parameters and hence will choose c such that, first of all N does not diverge, and the angular momentum vanishes. There is one remaining problem, though.

The flat spacetime has a non-vanishing angular momentum. Even if we set $c = 0$. In fact, if we calculate its angular momentum through the use of the killing vector χ^μ we find

$$J_{k=0} = -\frac{1}{8\pi}\int_{\partial M} \star d\chi = \infty \tag{5.25}$$

The case is even worse than that found in the spherical and hyperbolic cases where a simple setting of c to zero removes divergence. The main problem lies in the term that needs to vanish for the divergence to be cancelled. Let us first consider an integral over the boundary of constant r where we integrate over the disc. For a region where x goes from zero to L, the area is that of a disc, πL^2. To cancel the divergence, the term that needs to vanish takes the form

$$L^2(L^2 + 2c) \tag{5.26}$$

Not only does it not vanish for arbitrary values of c, there are no values of c for which it does. The reason here is that we cannot set c a priori to be the same value as L^2, since it is not precisely defined be a certain finite value. However, it is still possible to rid us of this divergence.

We rely on a similar trick to the one we did to cancel the divergence of the mass. Making use of the fact that ξ^μ is a killing vector, we find

$$\begin{aligned} d\psi &= \star\, \chi \\ (d\psi)_{\mu\nu\rho} &= \epsilon_{\lambda\mu\nu\rho} g^{\delta\lambda} \chi_\delta \end{aligned} \tag{5.27}$$

This can then be used to calculate a non-divergent angular momentum that does depend on the value of c. It is important to maintain that this does not mean to say that this angular momentum is not divergent. However, one can either choose to subtract an $m = 0$ background to find a non-divergent value or simply set $c = 0$ as we will do in our case. Thus, for $c = 0$

$$J_{k=0} = \frac{1}{8\pi} \int_{\partial M} \star d\chi + \psi = 0 \tag{5.28}$$

5.4 Action Calculation

The action, as we mentioned in chapter 3, is constituted of three different integrals. The Einstein-Hilbert-Maxwell term which is concerned with the bulk. The first boundary term, the Gibbons-Hawking one, treats the fact that the Ricci scalar R is a function is the second order derivatives of the metric. This is a problem because the path integral formalism used to calculate the temperature requires the variation to be on first derivatives at most. The last term is the counterterm, which is used to normalise the action as it diverges without its presence.

$$I = I_{\text{EH}} + I_{\text{GH}} + I_{\text{CT}} \tag{5.29}$$

Leading to

$$I = \frac{\beta}{2l^2 r_h}((nV + p)^2 + m r_h - V^2 r_h^2) l^2 - 3n^2 r_h^2 - r_h^4) \tag{5.30}$$

It is very important to note the type of ensemble we are in. Since we fix A_μ this entails that we will be fixing the magnetic charge at the horizon along with the electric potential. The electric charge can be fixed at the boundary by adding the following term to the action

$$I_Q = \frac{-1}{4\pi} \int_{\partial M} d^3x \sqrt{-h}\, n_a F^{ab} A_b \tag{5.31}$$

However, we will not be changing our ensemble from the mixed to the canonical one. This owes to the substitutions needed when we make the choice on the fixed parameters. Since we will be fixing the electric potential Φ_e, we will substitute for the terms that contain the electric charge q. This is to ensure that the action is written in terms of the thermodynamic variables that are fixed.

Working in the Canonical ensemble introduces terms that make finding an analytical solution for the critical points become much harder. The spherical case, treated through the canonical ensemble, was done recently,[40] but the solution for the critical points was only done perturbatively. It is important to note that their treatment differs from ours in many ways.

The integrals over the non-r boundaries introduce a scaling term to the action and the charges. As we have stated previously, we have chosen to set the weights to unity so that we can compare all the spacetimes depending on the parameter k. The values would not change the behaviour at all, which is our main interest.

The above expression for the action is subtly deceiving. There is no explicit dependence on k in the expression. However, there is a dependence which can be highlighted by examining the mass m, and periodicity β. We can see that below

$$\beta = \frac{4\pi r_h l^2 (r_h^2 + n^2)}{3(r_h^2 + n^2)^2 - l^2(p^2 + q^2 - k \cdot (r_h^2 + n^2))} \tag{5.32}$$

$$m = \frac{l^2\big(p^2 + q^2 + k \cdot (r_h^2 - n^2)\big) + r_h^4 + 6n^2 r_h^2 - 3n^4}{2 r_h l^2} \tag{5.33}$$

As we stated above, it's important to be working in terms of our thermodynamic parameters. We will be changing several of the variables, l, q, and p. We will substitute l because we will be varying the pressure. For the charges, it is dependent on the construction of the thermodynamic system itself. As we discussed above, it depends on what parameters we have fixed. The substitutions take the following form:

$$
\begin{aligned}
l &= \sqrt{\frac{3}{8\pi P}} \\
q &= \frac{n(p_h - 2n\Phi_e) + \Phi_e(r_h^2 + n^2)}{r_h} \\
p &= p_h - 2n\Phi_e
\end{aligned}
\tag{5.34}
$$

After making the appropriate substitutions we arrive at

$$
\beta = \frac{4\pi r^3}{-p_h^2 + kr^2 + 2np_hV + (n^2 + r^2)(8P\pi r^2 - V^2)} \tag{5.35}
$$

$$
m = \frac{a_6^{'} r_h^6 + a_4^{'} \cdot r_h^4 + a_2^{'} \cdot r_h^2 + a_0^{'}}{6r_h^3} \tag{5.36}
$$

Where the coefficients a_i are

$$
\begin{aligned}
a_6^{'} &= 8\pi P \\
a_4^{'} &= \left(48n^2\pi P + 3(\Phi_e^2 + k)\right) \\
a_2^{'} &= 3\left((p_h - n\Phi_e)^2 + n^2(\Phi_e^2 - k)\right) - 24n^4\pi P \\
a_0^{'} &= 3n^2(p_h - n\Phi_e)^2
\end{aligned}
\tag{5.37}
$$

One interesting feature of Taub-NUT-AdS is the possibility for an event horizon at positive r_h with negative spacetime mass. This happens mainly due to the $-3n^4$ term as it makes a larger contribution for small r_h. Other negative contributions may also exist, depending on the relative values of Φ_e and k. This will not be the last time their relative values dictate something about the behaviour of our system.

5.5 K-Dependent Thermodynamics

In this sub-chapter we explore the different thermodynamic systems for the three spacetimes. This will include checking whether it satisfies the Gibbs-Duhem relation, the first law, and the Smarr relation. In checking the thermodynamic potential we will take the appropriate derivatives with respect to our parameters and check them against the charges we calculated. Checking the first law will be done similarly. The Smarr, and Gibbs-Duhem relations are particular quantitative relationships between the parameters that need to be satisfied.

We begin by directly calculating the entropy of our spacetime. For the Taub-NUT-AdS spacetime, there is more than one way to approach this. Some treatments [41] treat the misner strings as having their own temperature and entropy. This is not the case in our treatment. The main avenues where other approaches differ from ours are the misner charge and potentials themselves.[42–44]The entropy in our formulation is the fourth of the surface area of the horizon, this follows along the treatments found here.[36, 37, 45] We calculate our entropy from the action directly through the equation presented below.

$$S = \partial_\beta I - I \tag{5.38}$$

Which evaluates to

$$S = \pi(r_h^2 + n^2) \tag{5.39}$$

The entropy turned out to be the same value we expected. We will now start checking the other thermodynamic properties. We start with the chemical potential associated with our partition function. As we stated in chapter 3, each partition function has an associated chemical potential. That would be the Helmholtz energy for the canonical ensemble or the grand canonical potential for the grand canonical ensemble. In our case we refer to it as Ω. We expect it to be a function in the parameters that are fixed, namely T, n, Φ_e, p_h, and P. The potential function can reached by dividing the action I by β. This owes to the following relation

$$Z_{mixed} = e^{-\beta\Omega} = e^{-I} \tag{5.40}$$

Expressing the relation between the potential and thermodynamic parameters differentially, we find

$$d\Omega = -SdT + \Phi_m dp_h + \Phi_n dn - qd\Phi_e + VdP \tag{5.41}$$

If we were interested in knowing the thermodynamic volume, for example, we would take the partial derivative of Ω with respect to P with all the other parameters held constant.

$$V = \left(\frac{\partial\Omega}{\partial P}\right)_{T,p_h,n,\Phi_e} \tag{5.42}$$

Which turns out to be

$$V = \frac{4}{3}(r_h^3 + 3r_h n^2) \tag{5.43}$$

This formulae for the volume is a hint that we are moving in the right track. Whenever we have a quantity that makes us different from the limiting case, we want to see that we reduce to the limiting case when the charge dies. If the nut charge dies, we get the Schwarzschild volume. For a general Taub-Nut spacetime, as n dies we get the corresponding spherically symmetric metric, be it a Schwarzchild or sner-Nordström black hole.

We now vary the potential function with respect to the fixed parameters to find their conjugate quantities

$$\begin{aligned} \left(\frac{\partial\Omega}{\partial\Phi_e}\right)_{T,p_h,N,P} &= -q \\ \left(\frac{\partial\Omega}{\partial p_h}\right)_{T,P,n,\Phi_e} &= \Phi_m \\ \left(\frac{\partial\Omega}{\partial T}\right)_{p_h,P,n,\Phi_e} &= -S \end{aligned} \tag{5.44}$$

Relying on n being fixed on boundary, we then calculate its conjugate thermodynamic potential

$$\left(\frac{\partial\Omega}{\partial n}\right)_{T,P,\Phi_e,p_h} = \Phi_n \tag{5.45}$$

$$\Phi_n = \frac{\left(\frac{\Phi_e^2}{2} - 4\pi P r_h^2\right)n^3 - p_h V n^2 + \left(3\Phi_e^2 - (k + r_h^2 - 4\pi P r_h^2)r_h^2 + p_h^2\right)\frac{n}{2} - p_h\Phi_e r_h^2}{r_h^3} \tag{5.46}$$

We then move on to the first law. We define the total energy of the spacetime as

$$U = \partial_\beta I + q\Phi_e \tag{5.47}$$

This results in

$$U = \frac{4\pi P r^4 + \frac{3}{2}\left(8\pi P n^2 + k + \Phi_e^2\right)r_h + \frac{3}{2}(p_h^2 - n^2\Phi_e^2)}{3\, r_h} \tag{5.48}$$

It is easy to show that we could have reached this from Ω through the use of legendre transformations. Below, we express the relation between differentials

$$dU = TdS + \Phi_m dp_h - nd\Phi_n + \Phi_e dq + VdP \tag{5.49}$$

We want to show as we did above that the relations hold when taking the partial derivative of the internal energy.

$$
\begin{aligned}
\left(\frac{\partial U}{\partial S}\right)_{q,p_h,\Phi_n,P} &= T, \qquad \left(\frac{\partial U}{\partial p_h}\right)_{S,q,\Phi_n,P} = \Phi_m \\
\left(\frac{\partial U}{\partial \Phi_n}\right)_{q,p_h,S,P} &= -N, \quad \left(\frac{\partial U}{\partial q}\right)_{S,p_h,\Phi_n,P} = \Phi_e \\
&\left(\frac{\partial U}{\partial P}\right)_{S,p_h,\Phi_n,q} = V
\end{aligned}
\tag{5.50}
$$

We will now construct our Smarr relation. Let us consider each of our parameters. We know that U varies as L, the entropy corresponds to an area and thus has L-dimension 2. Both q and p have dimension 1. The pressure has a dimension of -2 while Φ_n, being a potential, is dimensionless. This should result in the following Smarr relation

$$
1 \cdot U = 2 \cdot (TS - VP) + 1 \cdot (q\Phi_e + p_h\Phi_m) + 0 \cdot n\Phi_n \tag{5.51}
$$

It is important to note that the internal energy above is related to the mass M through the following relation

$$
U = M - 2n\Phi_n \tag{5.52}
$$

inserting that into the Smarr formula leads to

$$
M = 2(TS + n\Phi_n - VP) + q\Phi_e + p_h\Phi_m \tag{5.53}
$$

The last thermodynamic relation we wish to satisfy is the Gibbs-Duhem relation. It serves to describe the interactions between the different chemical potentials within a system. In our case it takes the form

$$
\frac{I}{\beta} = M - n\Phi_n - TS - q\Phi_e \tag{5.54}
$$

It is straightforward to show that the relation holds.

Chapter 6

Dyonic Taub-NUT Phase Structure

We will approach studying the phase transitions in these black holes as follows. We will first find out the phases we are able to consider for phase diagrams. The criteria will be that the black hole is thermally and mechanically stable. We will go over each of them briefly before examining the expressions for the temperature and the pressure. Subsequently, we take each of them and examine their behaviour as k varies.

To assess the thermal stability of the system we will be considering the isobaric heat capacity. Whenever it is positive we will have a thermally stable system, and unstable otherwise. The isothermal compressibility will be used in determining whether a system is mechanically stable or not.

After this general discussion we will provide examples for the transitions and critical behaviour for each different value of k. For each one we will talk about the kind of phase transitions that can occur and the phase structure.

6.1 Thermal Stability

The heat capacity is traditionally the amount of heat required to increase the temperature of a system. For a negative heat capacity taking away heat from the system would *increase* its temperature. To examine this, let us first recall the expression for the temperature

$$T = \frac{8\pi P r_h^4 + (8\pi P n^2 - \Phi_e^2 + k) r_h^2 - (p_h - n\Phi_e)^2}{4\pi r_h^3} \tag{6.1}$$

We will generally discuss the behaviour of the temperature as a function in r_h. How it acts at small and large values of r_h and how that reflects on the phase structure. We appeal to information about the number of roots of the first derivative and the behaviour of T at the boundaries. We can use this to fully describe the possible phases. We will then calculate the critical points that exist and the conditions for their existence.

We will separate the cases while describing the behaviour of the temperature with respect to r_h. There is a distinct case when $p_h = n\Phi_e$. In this case the constant term in the numerator vanishing, this affects the behaviour of T in the limit where r_h approaches zero. Whenever $p_h \neq n\Phi_e$ the constant term $(p_h - n\Phi_e)^2$ is always positive-definite, and the temperature approaches negative infinity. If this does not happen, the sign of the temperature in the limit of small r_h will depend on the r_h^2 coefficient. This will significantly change the behaviour of the temperature with respect to the horizon radius.

We will begin with the more general case in mind. The core of the analysis will pivot on the number of roots of the first derivative. We know that the leading term for an AdS spacetime, where $P > 0$, is always positive. This governs the asymptotic behaviour for T as a function in r_h and means that it will be a strictly increasing function as r_h becomes large. This is of value due to the nature of our entropy. In the previous chapter we calculated the entropy of our spacetime to be

$$S = \pi(r_h^2 + n^2) \tag{6.2}$$

Thus, it is always the case for positive r_h that

$$\left(\frac{\partial S}{\partial r_h}\right) > 0 \tag{6.3}$$

The specific heat c_P for constant pressure can be expressed as

$$c_P = \left(\frac{\partial S}{\partial r_h}\right)_P \left(\frac{\partial r_h}{\partial T}\right)_P \tag{6.4}$$

This leads to an easy identification of the heat capacity with the slope of the temperature with respect to r_h. Whenever the slope is positive, the heat capacity is positive, whenever it is negative the heat capacity is negative. Between r_h being zero, and large r_h, the number of roots of the first derivative will lead us into knowing the behaviour of the function. The first derivative takes the following form

$$\frac{\partial T}{\partial r_h} = \frac{8P\pi r_h^4 + r_h^2(-8n^2P\pi + \Phi_e^2 - k) + 3(p_h - n\Phi_e)^2}{4\pi r^4} \tag{6.5}$$

The numerator is an even function in r, this means that in the event that roots exist, we will have at most two positive roots. Since the asymptotic behaviour on the domain takes the function from approaching negative infinity to positive infinity, unless a root coincides with an inflection point, we must have an even number of roots. Thus, if there is only one positive root then it will also be an inflection point.

Let us first examine the case where we have two positive roots. Since the temperature moves asymptotically from a negative to a positive value, the slope will change from positive to negative at the first root of the derivative. This will be a local maxima. The next root will then arrive at a local minima. The region between both roots will always have a negative slope, making it always unstable. Since we cannot have more than two positive roots, this tells us that we will have at most two stable phases.

Any two stable phases will be seperated by a region where no stable black hole can exist. By virtue of a local maxima preceding a local minima on a function whose asymptotic behaviour is increasing, taking a horizontal line at constant temperature between the maxima and minima will intersect the curve at three separate radii. The smallest one will have a positive slope and will hence be stable, as will the largest. The middle one will always have a negative heat capacity.

The two stable phases will exist at different radii. It's important to note that the entropy changes as the radius changes. This means that between the larger and smaller black hole, there is a discontinuity in entropy. This discontinuity in entropy signifies the existence of a first order phase transition. To examine the behaviour at a deeper level, let us take a look at the second derivative

$$\frac{\partial^2 T}{\partial r_h^2} = \frac{r_h^2(k + 8n^2P\pi - \Phi_e^2) - 6(p_h - n\Phi_e)^2}{2\pi r^5} \tag{6.6}$$

The second derivative has at most one positive root for the same pressure. Since we cannot have an odd number of first derivative roots unless the root is an inflection point, this means that we have at most one of such. However, this doesn't mean that there is only one critical pressure. This will become clearer in our discussion of mechanical stability. Whenever this happens, we will have a critical point. A critical point occurs when

$$\frac{\partial T}{\partial r_h} = \frac{\partial^2 T}{\partial r_h^2} = 0 \tag{6.7}$$

The possible positive roots of the first derivative are

$$r_h = \frac{1}{4}\sqrt{8n^2 + \frac{k}{P\pi} - \frac{\Phi_e^2}{P\pi} \pm \frac{\sqrt{(\Phi_e^2 - k - 8n^2 P\pi)^2 - 96P\pi(p_h - n\Phi_e)^2}}{P\pi}} \tag{6.8}$$

The critical point occurs at

$$r_h = \frac{1}{4}\sqrt{\frac{(8n^2 P\pi + k - \Phi_e^2)}{P\pi}} \tag{6.9}$$

It is easy to see that the radius at which the critical point occurs is the same as the roots whenever the portion after the $\pm$ sign is zero. Thus, we have a region that expands or contracts until it disappears completely. At the point where it just disappears, the coinciding roots signify the merging of the largest and smallest radii. At this point a second order phase transition occurs; there is no discontinuity in the entropy.

By solving for the case where the second square root vanishes, we can examine the values at which the critical point occurs. The condition for the critical point gives two independent equations, in principle, we can use that to parameterise two of the unknowns in term of the other variables. We use that to solve for the critical radius and the critical pressure. We can then simply plug these expressions into the equation for the temperature to get its critical value. We show each critical value below

$$\begin{aligned}
T_c &= \frac{kr_c^2 + 8\pi n^2 P_c r_c^2 - n^2\Phi_e^2 + 2np_h\Phi_e + 8\pi P_c r_c^4 - {p_h}^2 - r_c^2\Phi_e^2}{4\pi r_c^3} \\
r_c &= \frac{\sqrt{k + 8\pi n^2 P_c - \Phi_e^2}}{4\sqrt{\pi}\sqrt{P_c}} \\
P_c &= \frac{a_c + \sqrt{b_c}}{128\pi^2 n^4} \\
a_c &= 96\pi {p_h}^2 - 16\pi k n^2 + 112\pi n^2\Phi_e^2 - 192\pi n p_h \Phi_e \\
b_c &= (a_c)^2 - 256\pi^2 n^4(k^2 - 2k\Phi_e^2 + \Phi_e^4)
\end{aligned} \tag{6.10}$$

Though, we have been discussing the $p_h \neq n\Phi_e$ case, none of our calculations are contingent on this condition. In this case the temperature will no longer always approach negative infinity as r_h approaches zero. This is due to the positive definite term vanishing and the behaviour being dominated by the r_h^2 term. When this term is negative, the asymptotic behaviour will approach negative infinity and when the term is positive it will approach positive infinity.

This then results in Hawking-Page-like behaviour whenever the term is positive. As P changes, however, a transitions occurs. There is no critical point for this transition, though. For the vanishing of the second derivative in this case the behaviour must be independent of r_h. This means that both lower order terms vanish for the first derivative. Since P is positive-definite, this means that there is no way for the first derivative to vanish. In fact it then becomes a straight line starting at the origin.

In this case, when $P < P_c$ we get Hawking-Page-like behaviour, $P = P_c$ results in completely linear behaviour and a naked singularity, and for $P > P_c$ we get a black hole that is everywhere stable. This is very similar to the variation with the charges in the sner-Nordström-AdS case.

6.2 Mechanical Stability

The compressibility is the instantaneous change in volume under the variation of pressure. For a system with positive compressibility, when there is an isothermal increase in pressure, there will be a decrease in volume. In contrast, a system with negative compressibility will see its volume increase under an increase in pressure. We use this criteria to find the stable cases where compressibility is positive. We can begin by examining the pressure

$$P = \frac{4\pi r_h^3 T + r_h^2(\Phi_e^2 - k) + (p_h - n\Phi_e)^2}{8\pi r^2(n^2 + r_h^2)} \tag{6.11}$$

We will follow in the steps of our analysis for the temperature. The behaviour is similar in the sense that we can see that whether $p_h = n \cdot \Phi_e$ or not will influence the asymptotic behaviour of the pressure in the limit of small r_h. We will be considering the more general case first before seeing how matters differ for this case. Let us first consider the isothermal compressibility

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \tag{6.12}$$

We will appeal to a similar trick to the one we used with the thermal capacity. Lets us consider the volume and how it changes with the temperature. The partial derivative for V with respect to r_h will always be positive definite whenever $r_h > 0$. This allows us to consider

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial r_h}\right)_T \left(\frac{\partial r_h}{\partial P}\right)_T \tag{6.13}$$

Thus, whenever the first derivative for the pressure with respect to r_h is negative, we have positive compressibility. When the lowest order term is positive definite, we then have that P approaches positive infinity for small r_h. I all cases, P will be approaching zero from above in the limit where r_h tends to infinity. This is because the temperature being positive definite.

We will proceed with calculating the first and second derivatives of the pressure.

$$\begin{aligned}\frac{\partial P}{\partial r_h} &= \frac{-2\pi r_h^5 T + r_h^4(k-\Phi_e^2) + 2n^2\pi r_h^3 T - 2r_h^2(p_h - n\Phi_e)^2 - n^2(p_h - n\Phi_e)^2}{4\pi r_h^3(n^2+r_h^2)^2}\\ \frac{\partial^2 P}{\partial r_h^2} &= \frac{4\pi r_h^7 T + 3r_h^6(\Phi_e^2 - k) - 12n^2\pi r_h^5 T + r_h^4(kn^2 + 10p_h^2 - 20np_h\Phi_e + 9n^2\Phi_e^2)}{4\pi r_h^4(n^2+r_h^2)^3}\\ &+ \frac{r_h^2(9n^2(p_h - n\Phi_e)^2) + 3n^4(p_h - n\Phi_e)^2}{4\pi r_h^4(n^2+r_h^2)^3}\end{aligned} \tag{6.14}$$

The first derivative reinforces the analysis we did above, for low r_h the compressibility is always positive provided that $p_h \neq n\Phi_e$. It would be the converse in the case that it is. We cannot explicitly solve for the roots of the equation since it is a fifth degree polynomial with arbitrary coefficients. What we can do, is check if we can solve both the above equations together for the critical point(s). In practice, a slight variation around the critical point leads us to the roots of the first derivative.

It is interesting to note the difference between the expressions above and those for the derivatives of the temperature. We can no longer ascertain propositions so simply. The equations above do not allow us to easily claim the number of critical points as we previously did. The behaviour is richer and more complex. We move on to calculating the critical points through

$$\frac{\partial P}{\partial r_h} = \frac{\partial^2 P}{\partial r_h^2} = 0 \tag{6.15}$$

The critical points occur at

$$
\begin{aligned}
P_c &= \frac{T_c r_c + (\Phi_e^2 - k)}{2(n^2 + r_c^2)} + \frac{(q - n\Phi_e)^2}{8\pi r_c^2 (n^2 + r_c^2)} \\
T_c &= \frac{2(p_h - n\Phi_e)^2}{\pi r_c^3} \\
r_c &= \sqrt{\frac{6(p_h - n\Phi_e)^2 \pm \sqrt{6(p_h - n\Phi_e)^2 - 12n^2(k - \Phi_e^2)(p_h - n\Phi_e)^2}}{2(k - \Phi_e^2)}}
\end{aligned}
\tag{6.16}
$$

As we had stated above, it is possible that we have more than one critical point. We manipulate the expression for the critical radius to find the cases in which it is real. This can allow us to establish the conditions for the existence or non-existence of critical behaviour. This highlights the usefulness of the use of k in our analysis. We can now tell how the existence of critical points depends on horizon geometry.

We separate treat each possible critical radius separately. Let us denote

$$
\begin{aligned}
r_{c_1} &= \sqrt{\frac{6(p_h - n\Phi_e)^2 + \sqrt{6(p_h - n\Phi_e)^2 - 12n^2(k - \Phi_e^2)(p_h - n\Phi_e)^2}}{2(k - \Phi_e^2)}} \\
r_{c_2} &= \sqrt{\frac{6(p_h - n\Phi_e)^2 - \sqrt{6(p_h - n\Phi_e)^2 - 12n^2(k - \Phi_e^2)(p_h - n\Phi_e)^2}}{2(k - \Phi_e^2)}}
\end{aligned}
\tag{6.17}
$$

For r_{c_1}, the condition for existence is

$$
\left(\left(p_h \leqslant n\Phi_e - \frac{\sqrt{kn^2 - n^2\Phi_e^2}}{\sqrt{3}}\right) \vee \left(p_h \geqslant n\Phi_e + \frac{\sqrt{kn^2 - n^2\Phi_e^2}}{\sqrt{3}}\right) \right) \wedge \left(\Phi_e^2 < k\right)
\tag{6.18}
$$

While for r_{c_2} there are two possibilities for existence. Whenever any of the below situations occur, r_{c_2} will exist

$$
\left(\left(p_h \leqslant n\Phi_e - \frac{\sqrt{kn^2 - n^2\Phi_e^2}}{\sqrt{3}}\right) \vee \left(p_h \geqslant n\Phi_e + \frac{\sqrt{kn^2 - n^2\Phi_e^2}}{\sqrt{3}}\right) \right) \wedge \left(\Phi_e^2 < k\right)
\tag{6.19}
$$

$$
(k < \Phi_e^2) \wedge (p_h \neq n\Phi_e)
\tag{6.20}
$$

It is now clear that there are two distinct cases which depend on the relative values of k and Φ_e^2. Whenever k is less than Φ_e^2 we will have one critical point so long as $p_h \neq n\Phi_e$. However in the case where k is greater than Φ_e^2, we could have either have two critical points, one critical point, or none, depending on the value of p_h.

The above conditions allow us to know what we are looking for when we are exploring the phase structures of the different spacetimes. For example, r_{c_1} only exists for the spherical case since it is impossible for Φ_e^2 to be less than zero or negative one. However, this guarantees the existence of a critical point whenever $p_h \neq n\Phi_e$. In contrast the existence of critical points is not guaranteed for $\Phi_e < 1$.

6.3 Stability and Phase Structure For Hyperbolic Horizon Geometry

For the hyperbolic geometry, we have that $k = -1$. We will examine the existing phases for the hyperbolic geometry by considering the available phases. Upon finding the stable cases, we will find which phase is preferable.

Below, we plot the case where the pressure is just above P_c

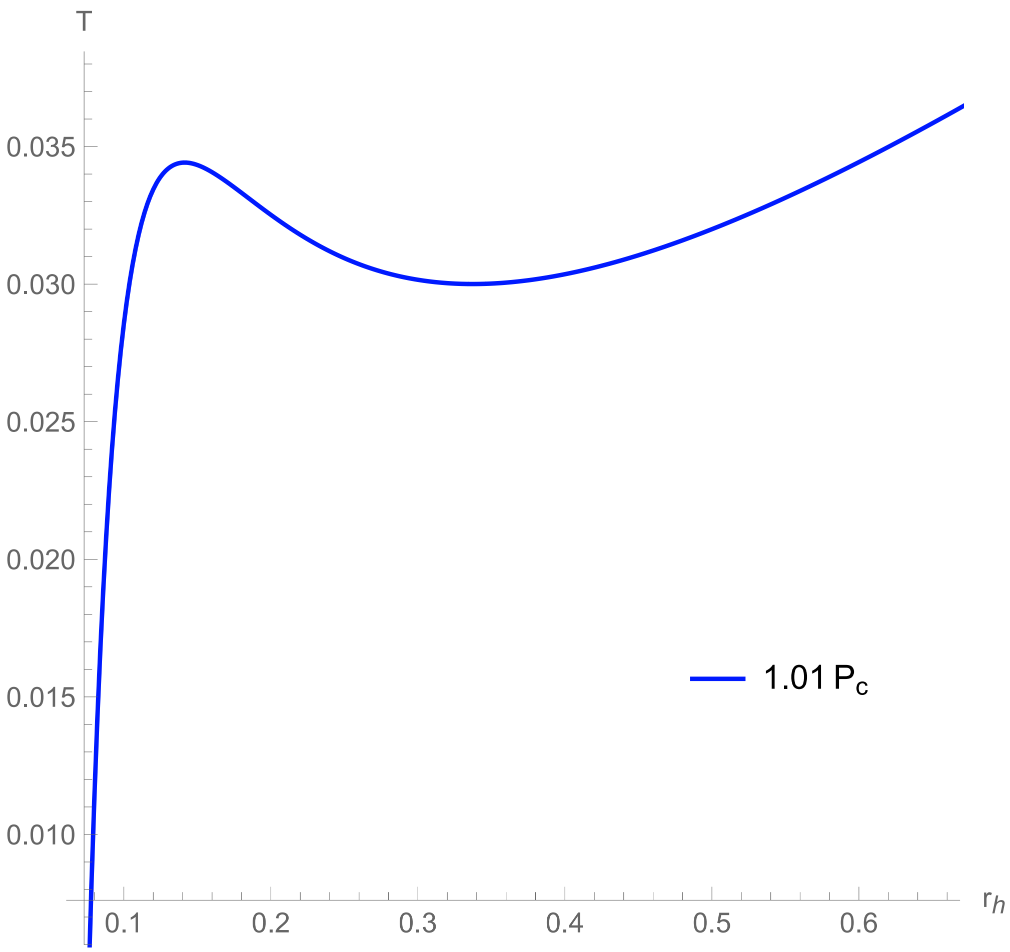

Figure 6.1: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 1, n = 1.98, p_h = 2, P_c = 0.0281$

Above the critical pressure we have a region where the slope is negative. This region is physically unstable. This means that we will have one stable region to its left, and one to its right. Between the local minima and the local maxima, there is a region where the black hole can stably exist at two different radii, but at the same temperature.

If we were to decrease the pressure until we reach the critical pressure, we will see that this behaviour will cease to exist.

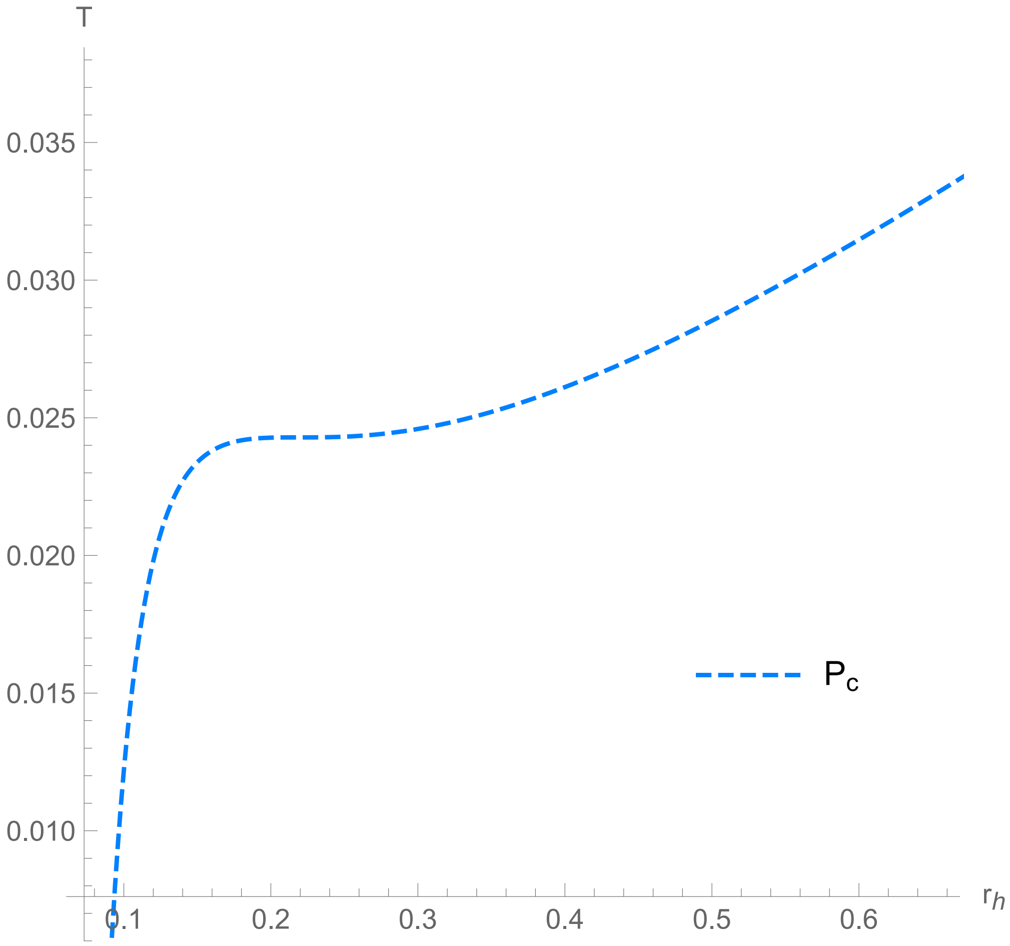

Figure 6.2: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1, n = 1.98, p_h = 2, P_c = 0.0281$

When we decrease it further, the graph continues to be monotonic. This is the only critical point that exists for the hyperbolic geometry as we found out above in our analysis for the mechanical stability. Only one critical point exists for the hyperbolic and flat geometries.

Above the critical pressure we find that a first order phase transition occurs. Below there transitions are second order. Since the function becomes monotonic, no two radii share the same temperature and there are no unstable regions. This then shows us that there is no discontinuity in the entropy and there is no region where a first order phase transition occurs.

When we plot the temperature with the radius, varying the pressure around the critical value we get

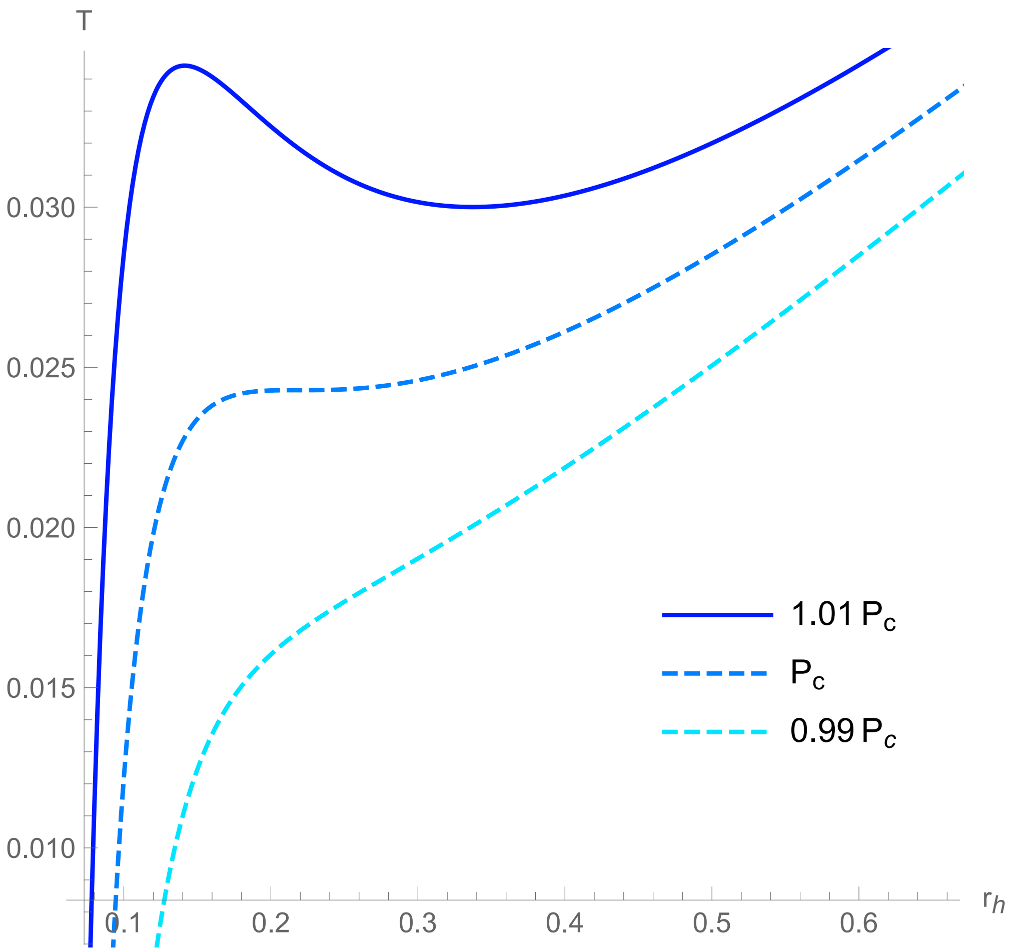

Figure 6.3: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1, n = 1.98, p_h = 2, P_c = 0.0281$

We now move on to plotting the pressure versus the horizon radius. As opposed to the temperature graph where the positive slope indicated stability, the case here is the opposite. In regions where the slope is negative, the phase will be a stable one. Whenever it is positive, the phase is not physical.

The graphs above were all isobaric. An isobaric process is one that occurs at constant pressure. The ones we will be examining now are isothermal. An isothermal process is a process where the temperature remains fixed. We begin by showing the existence of a region where there is a phase transition. We plot the pressure above the critical temperature below

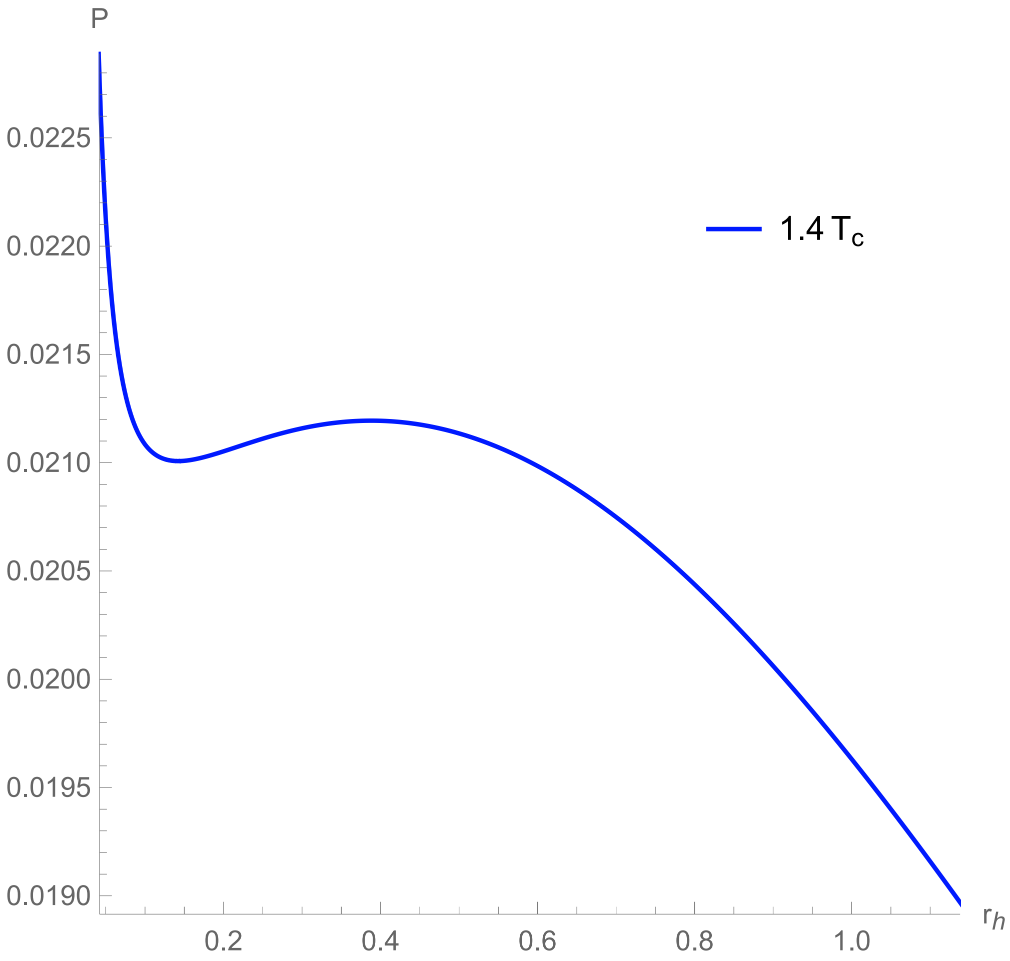

Figure 6.4: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1, n = 1.98, p_h = 2, T_c = 0.0243$

We can clearly see that there is a region where the black hole is unstable. This region is then replaced using the Maxwell equal area law. This entails that a line that crosses the unstable region with an area bound between itself and the graph, will intersect at two different radii on either side. When the areas on either side of the line and the graph are equal, these radii are the physical ones at which the transition occurs.

As we have seen before in the temperature case, we will examine the effect of altering the isotherm so that we cross the threshold of the critical temperature. We then get this behaviour

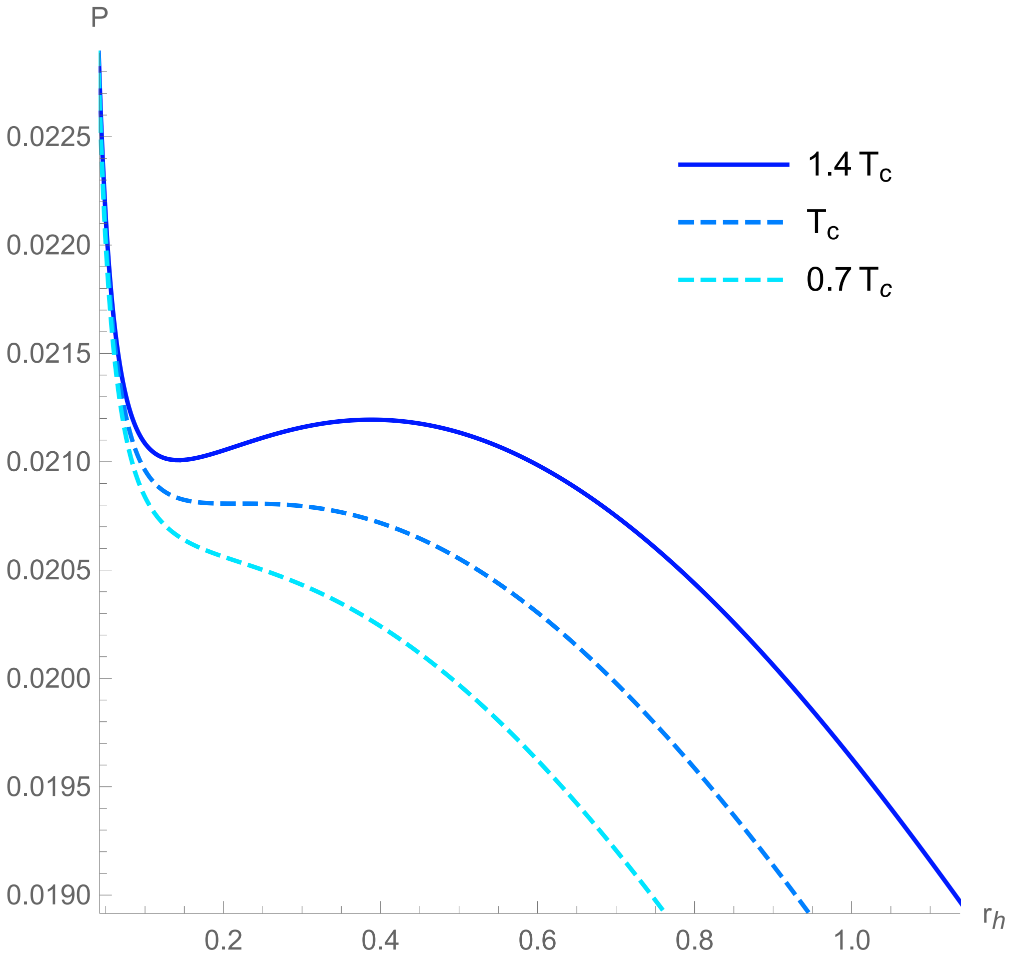

Figure 6.5: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 1, n = 1.98, p_h = 2, T_c = 0.0243$

The behaviour then becomes monotonous as we cross into the region where $T < T_c$. To find out which of the phases, the small or the large radius is more stable we will examine the variation with the chemical potential.

When the chemical potential is smaller we will have a region that is more stable than the other. Some regions may be stable in the sense that they have positive heat capacity and compressibility, however, that does not mean that they are preferred. To figure out which one is preferred we plot the potential with the pressure.

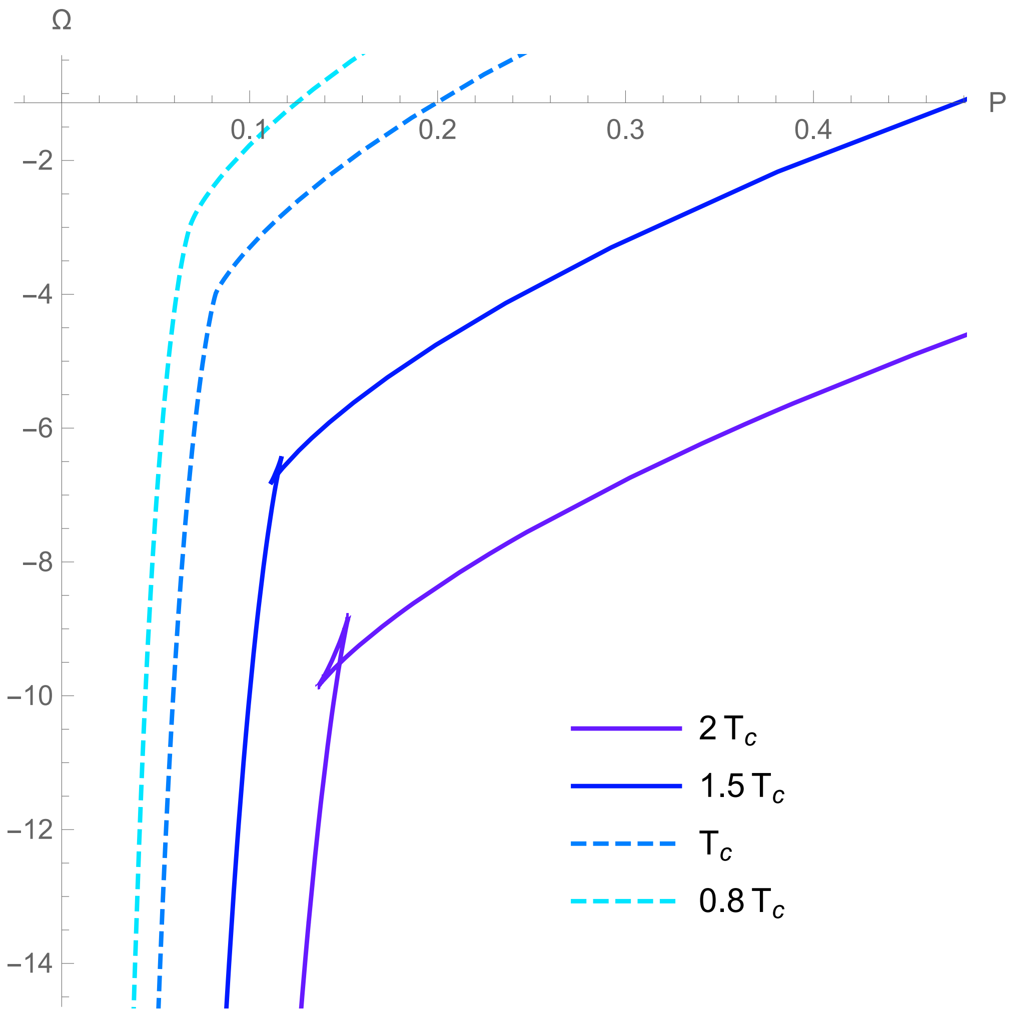

Figure 6.6: Isobaric plot for the chemical potential with respect to change in the horizon radius. $\Phi_e = 0.3$, $n = 1.98$, $p_h = 2$, $T_c = 0.570$

Instead of using the Maxwell equal area law directly, we make use of the fact that the two stable phases have the same chemical potential and lie on the same isotherm. This then allows us to solve for the radii in terms of our variables. Once that is the case we can the substitute in the term for the pressure or the temperature to plot the phase diagram.

The phase diagram shows us where there is a homogeneous transition and where the transition is first order. Any path that cuts a solid line in the phase diagram contains a first order phase transition. This means that in passing this line, the system endures a discontinuity in its entropy. We plot the phase diagram for the hyperbolic case below.

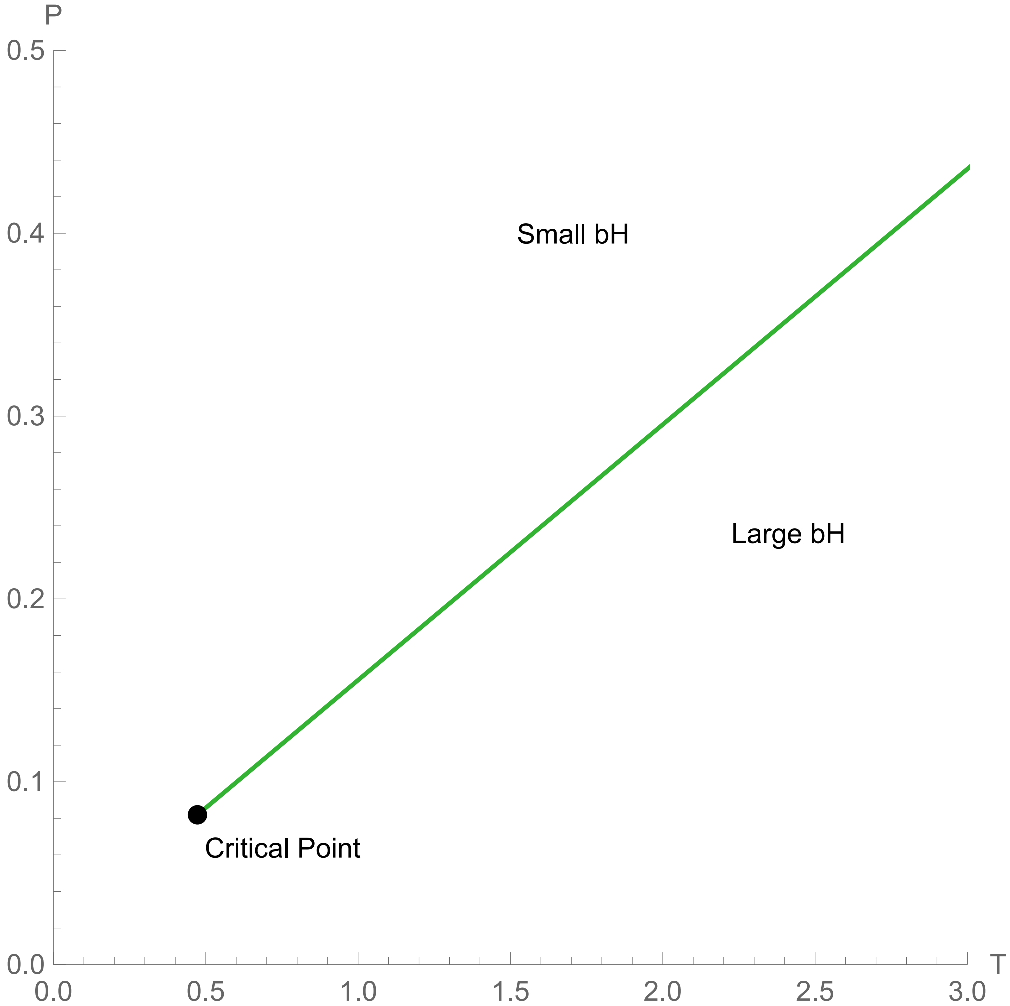

Figure 6.7: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1.5$, $n = 1.9$, $p_h = 2$, $P_c = 0.0820$, $T_c = 0.473$

With this, we conclude our discussion of the hyperbolic geometry phase structure. We will visit these results later in the conclusion to compare the different geometries in retrospect.

6.4 Stability and Phase Structure For Flat Horizon Geometry

For $k = 0$ the phase structure is almost identical. The only difference is that for $\Phi_e = 0$ there is no critical point for the flat case whilst there is one for the hyperbolic. This could be useful if we were interested in pursuing a canonical ensemble with $\Phi_e = 0$. Otherwise, the phase structure is completely identical.

As we did before, we show the case where we have a first order phase transition as $P > P_c$

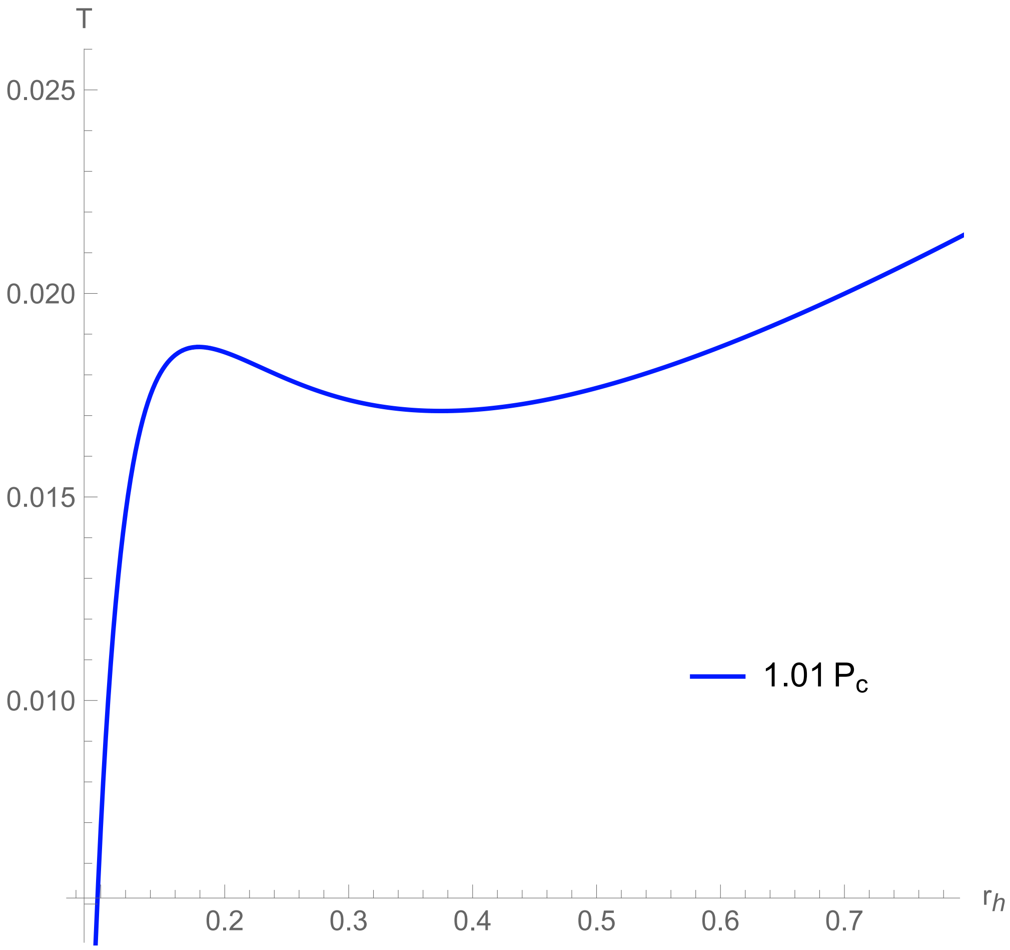

Figure 6.8: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 1, n = 1.98, p_h = 2, P_c = 0.0105$

In the region where there exists more than one radius for the same temperature, we have a first order phase transition. The discontinuity in entropy is the same as the flat case and signifies the phase transition.

Upon reaching the critical pressure the behaviour changes to

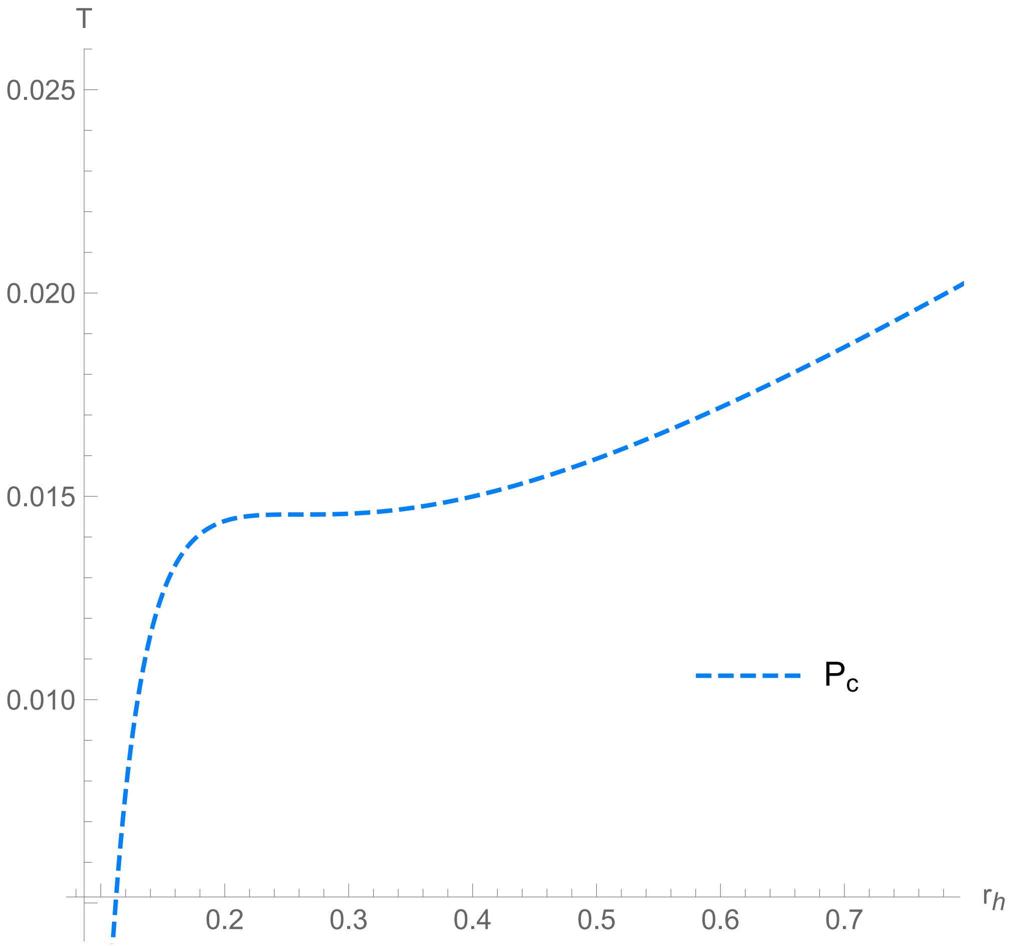

Figure 6.9: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1, n = 1.98, p_h = 2, P_c = 0.0105$

As we found while exploring the mechanical stability, there is only one critical point for the flat geometry. Thus, after decreasing the pressure to and beyond the critical point, the behaviour becomes completely monotonous. This indicates that there is a region below the critical pressure where a second order phase transition takes place.

Transitions below, and at the critical pressure, are second order but are first order above it. The lack of unstable region coupled with the fact that each black hole radius corresponds to a unique temperature is what shows us that the transition is second order.

We show the change in behaviour under a slight change in P around the critical point below

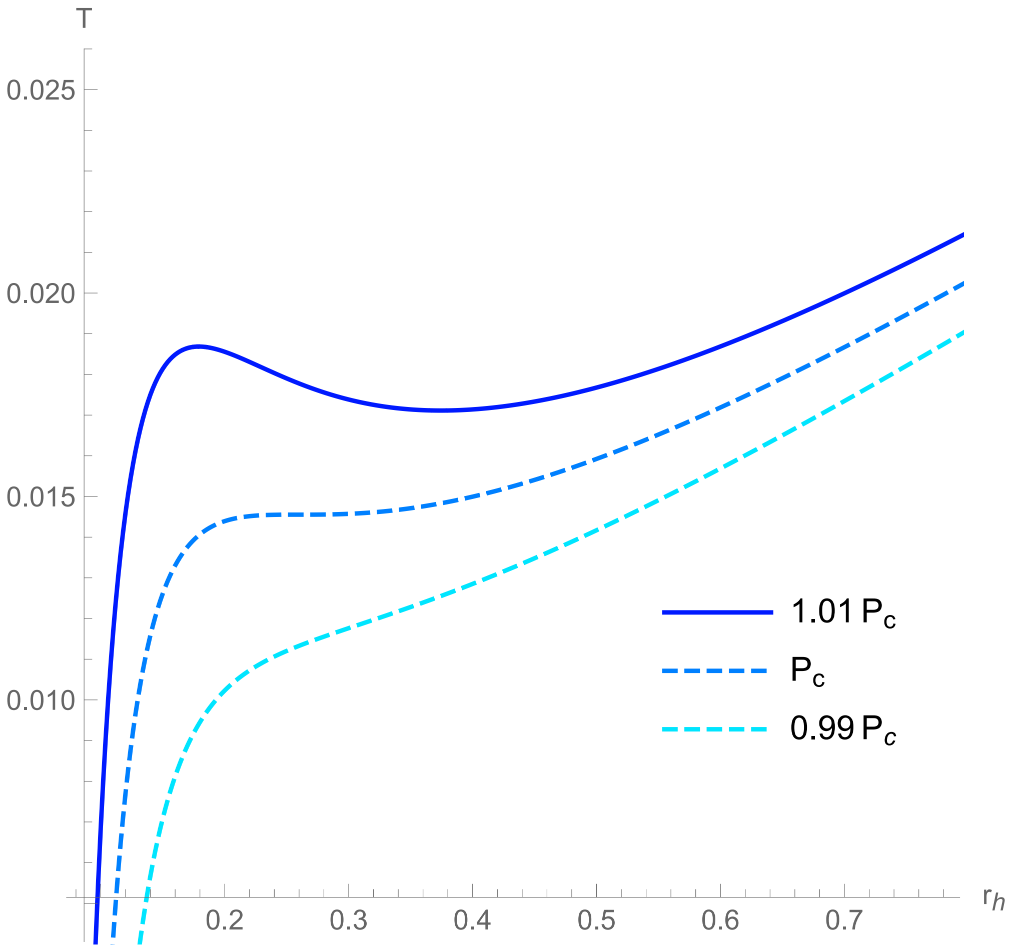

Figure 6.10: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1, n = 1.98, p_h = 2, P_c = 0.0105$

We can now do similar analysis in the $P - r_h$ plane. We will show the variation around the critical temperature as we did in the hyperbolic case. The change of the the behaviour for the pressure as the temperature changes is very similar to what we have seen above.

The main difference between the temperature and the pressure in the graphs is that the negative gradient region is the one that corresponds to a stable phase. This stands in contrast with the temperature graph. This owes to the difference between the conditions for stability. For the compressibility to be positive, the change of the pressure with respect to the volume must be negative.

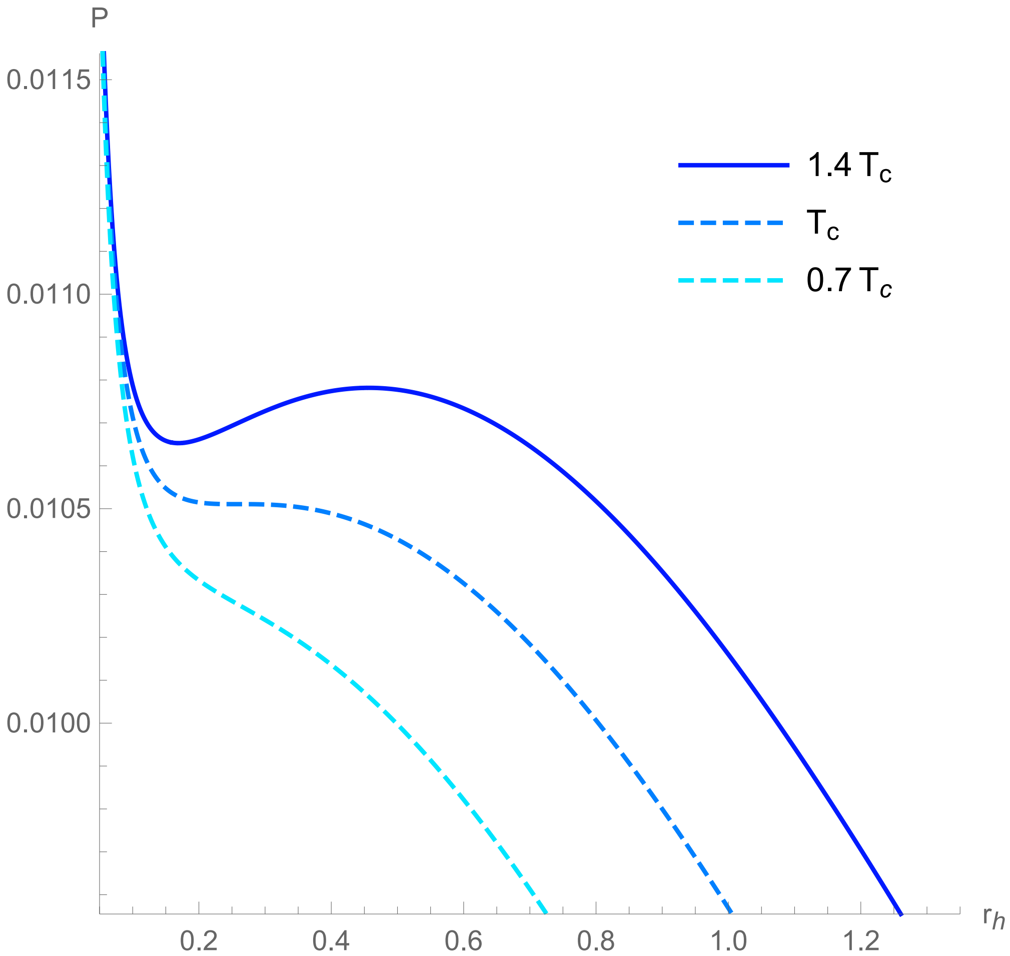

Figure 6.11: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1, n = 1.98, p_h = 2, T_c = 0.0145$

When the temperature increases beyond the critical temperature, a first order phase transition occurs. As with the temperature, the region in the middle is not physical. The physical radii can be found by employing the Maxwell equal area law or matching the chemical potential at both points. The question we ask now is which region will be preferred to the other, the larger or the smaller radius?

To answer this question we plot the variation of the chemical potential with the pressure, bearing in mind that a lower energy implies a preferred state. It's also important to not while examining the graph that a larger pressure corresponds to a smaller radius and vice versa. This may make the graph seem misleading at first glance.

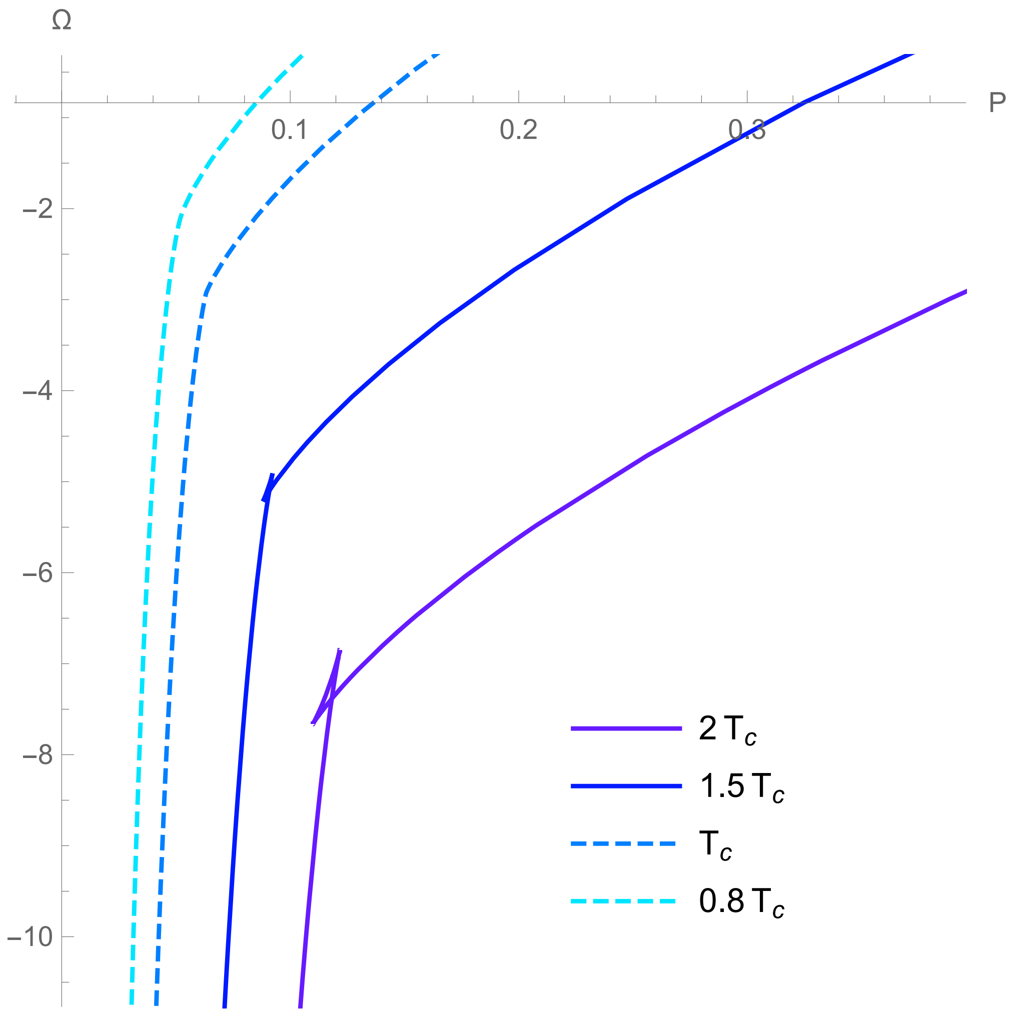

Figure 6.12: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 0.3$, $n = 1.9$, $p_{h} = 2$, $T_c = 0.469$

Below the critical temperature, there is no swallow tail behaviour in the graph. The swallow tail only begins to show when exceed the critical temperature and continues on in increasing. Below the critical temperature there is not discontinuity in the entropy and there is only a second order phase transition.

The point where both the lines representing the chemical potential meet is the one where the phase transition occurs. The transition happens at a fixed energy, and temperature. This means that the transition happens between two different black hole radii. This is what creates the discontinuity in the entropy.

We can now move on to constructing the phase diagram for the flat geometry.

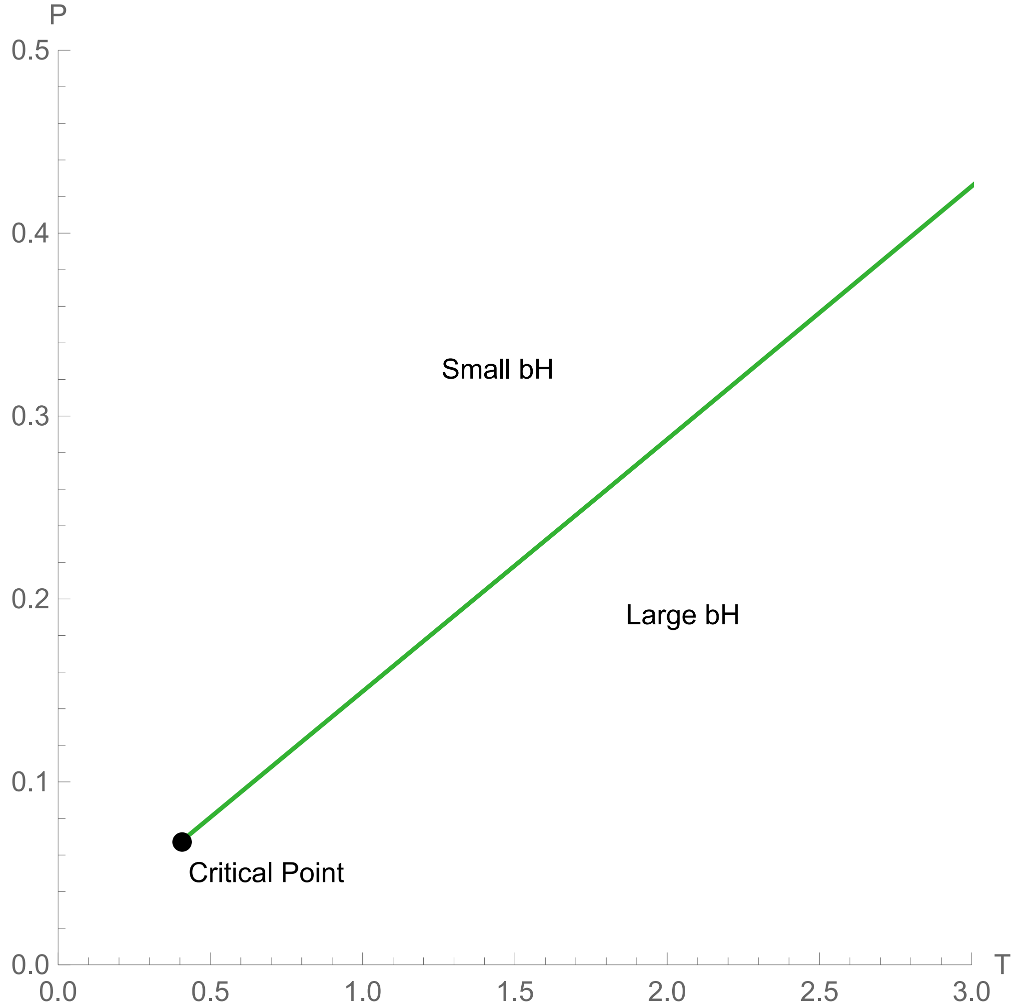

Figure 6.13: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 1.5, n = 1.9, p_h = 2, P_c = 0.0671, T_c = 0.407$

As is apparent from the work above, the first order transitions occurs above both the critical temperature, and pressure. This means that a black hole whose change in temperature or pressure allows it to move on a path that crosses the line, will see a discontinuity in its entropy. Another black hole, taking a path that dips below the critical point and then moves from one side to of the line to the other will not.

6.5 Stability and Phase Structure For Spherical Horizon Geometry

The spherical geometry corresponds to $k = 1$ and represents much richer phase structure than either the flat or the hyperbolic cases. This owes to the possibility of two critical points for some values of Φ_e. For others, the behaviours mimics that of the flat and hyperbolic cases. We will see the similar case first, taking $\Phi_e = 1.1$

Below, we plot the case where the pressure is just above P_c

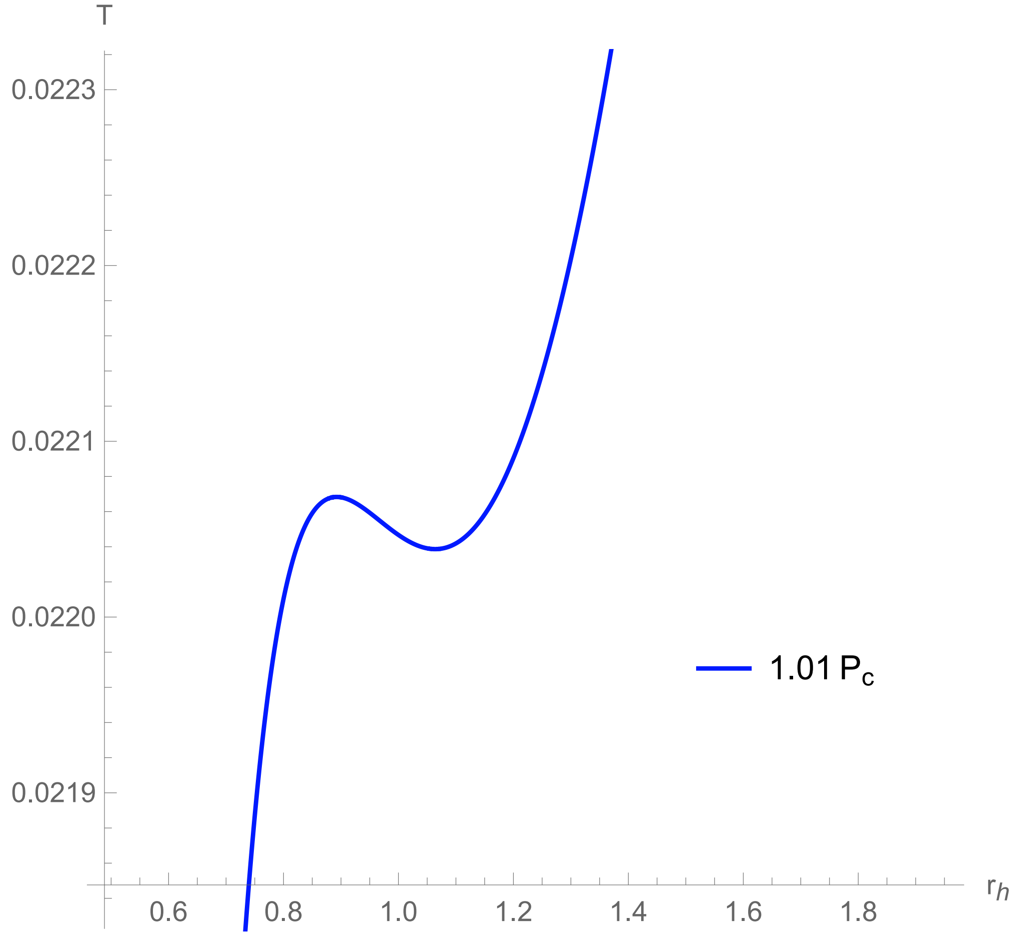

Figure 6.14: Isobaric plot for the temperature with respect to change in the horizon radius. $P > Pc$
$\Phi_e = 1.1, n = 1.98, p_h = 2, P_c = 0.00415$

Th region above the critical pressure contains the first order phase transition. This can be seen through the discontinuity in entropy between the existent phases. The stable cases are seperated by an unstable region thus the radii the black hole can exist at will have different radii which results in a discontinuity in entropy.

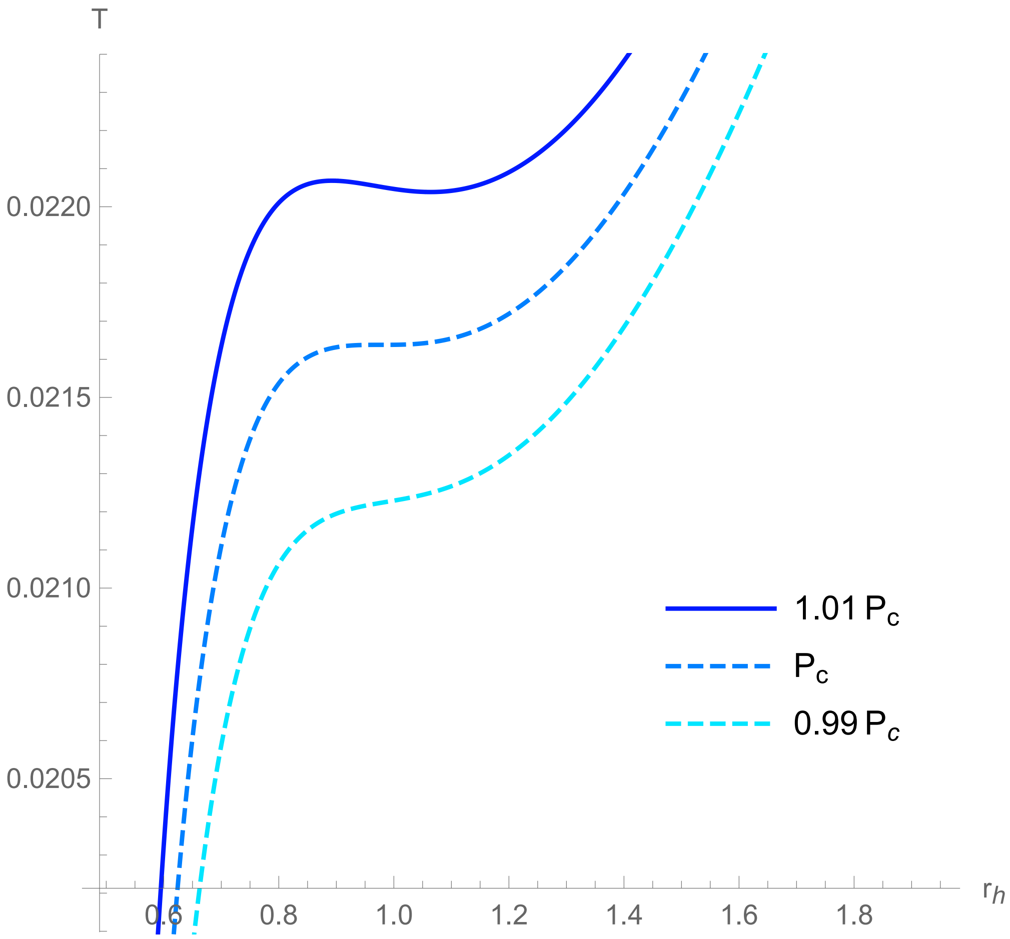

Figure 6.15: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 1.1$, $n = 1.98$, $p_h = 2$, $P_c = 0.00415$

The behaviour remains one to one for positive r_h at pressures below the critical point. This is only the case for the spherical geometry whenever the electric potential isn't less than one. This means that no matter how we decrease the pressure, the behaviour will remain monotonic.

In the monotonic region the black hole undergoes a continuous, second order phase transition as there is only one black hole radius for every temperature. This stands in contrast with the region were there is such a jump in the radius of the stable regions. In this region a first order transition takes place.

What happens above is the same as what happens whenever $\Phi_e^2 > k$. This is not unique to the spherical case. What are, however, are the kinds of phase transitions that can occur when $\Phi_e^2 < k$. This allows for two critical radii, quite different from the other cases.

We will then have a smaller and a larger critical radius. We have to wonder whether the phase structure will look anything like it does above. Will the first order transition occur above or below the critical temperature? We plot the transition at the smaller of the two radii below

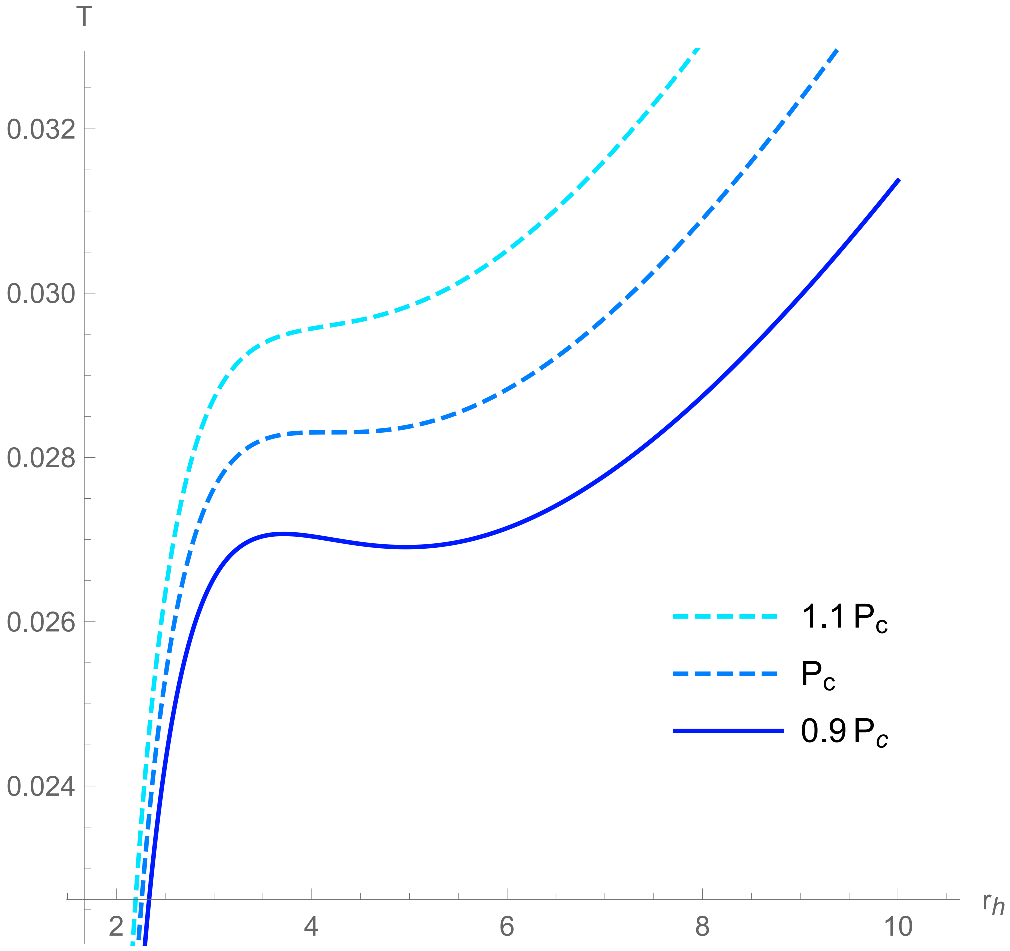

Figure 6.16: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 0.1$, $n = 1.98$, $p_h = 2$, $P_c = 0.00127$

The transition actually occurs below the critical pressure. That means that below the critical pressure we will always have a first order phase transition. This owes to the fact that this is the smaller of the two radii and that no critical points exist below it. We examine the behaviour at the larger radius to find

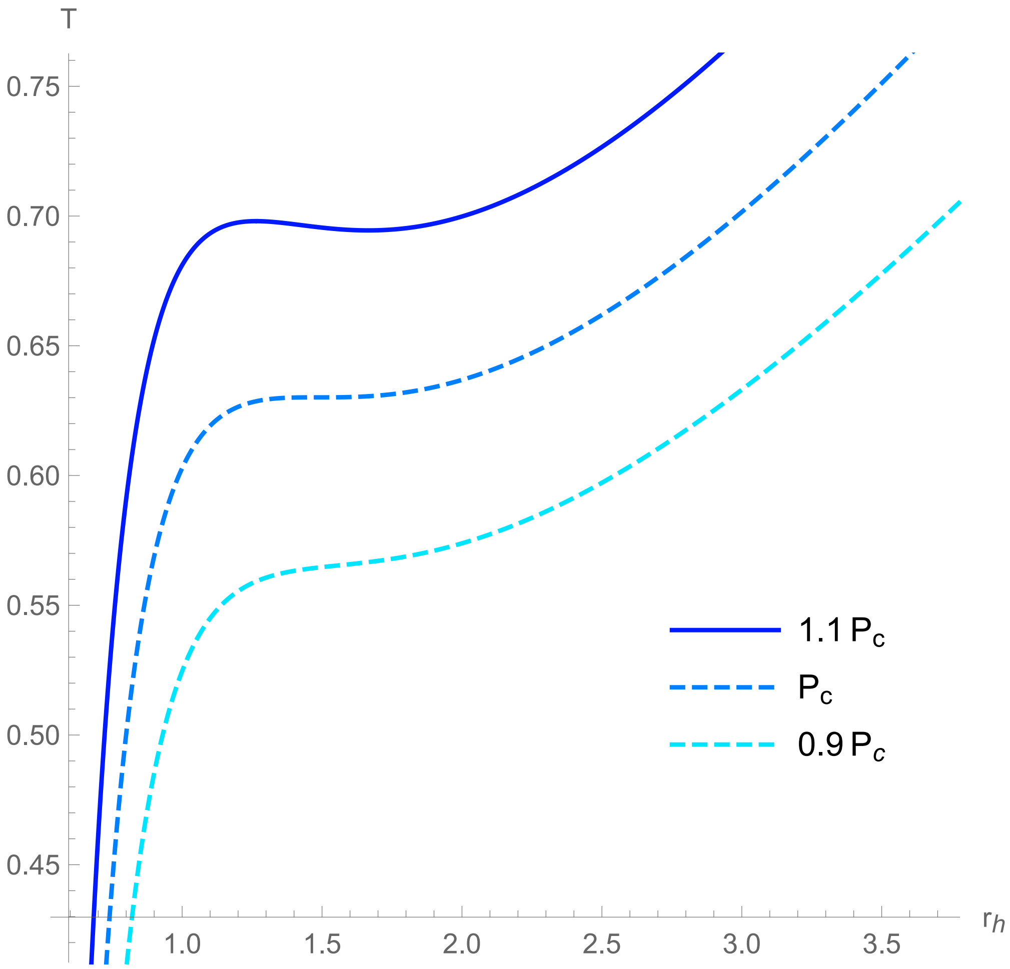

Figure 6.17: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 0.1$, $n = 1.98$, $p_h = 2$, $P_c = 0.0795$

This allows us to understand the phase structure of the spacetime better. It is not different from the other two for large P and T, a first order phase transition is inevitable beyond a certain point. We will now take a look at what happens when we consider the pressure under variation in r_h

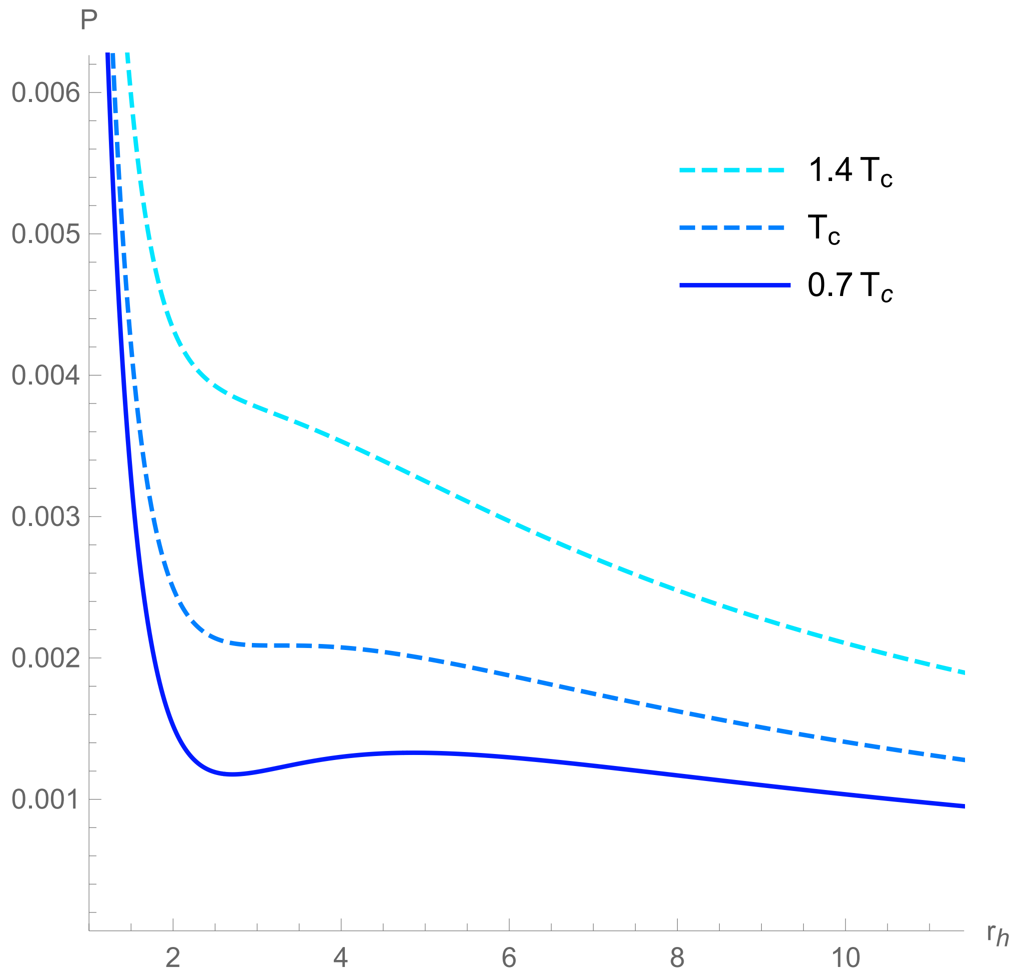

Figure 6.18: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 0.3$, $n = 1.98$, $p_h = 2$, $T_c = 0.0363$

As with the case in the temperature for the smaller critical radius, the first order transition occurs *below* the critical temperature and pressure. This entails that there is a region between both critical points where there is a smooth second order phase transition.

Now, let us consider the stability of the phases, since the larger black hole is the one with the larger entropy we would expect it to be the more stable one. This would also match the cases in the flat and the spherical horizons where the larger of the two black holes was preferred. To do so, we examine how the chemical potential changes under variation in P

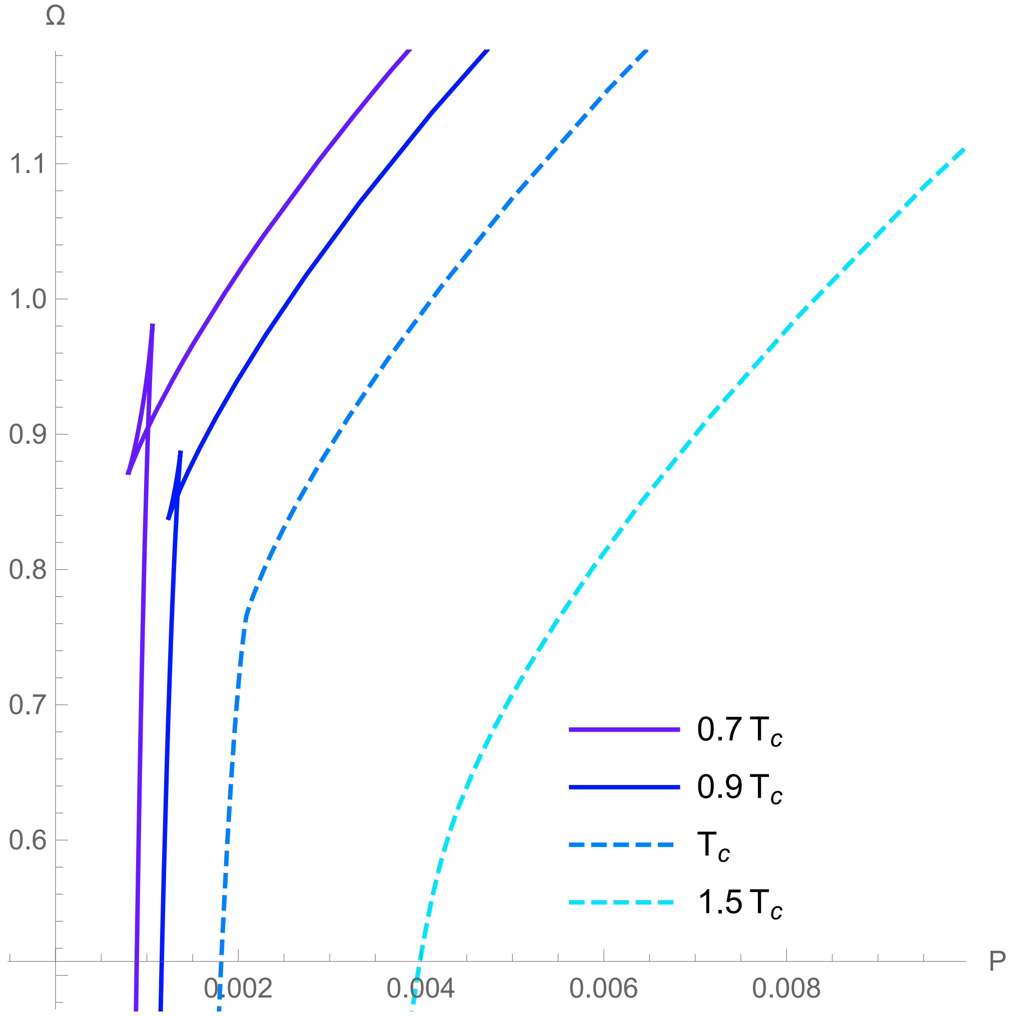

Figure 6.19: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 0.3$, $n = 1.98$, $p_h = 2$, $T_c = 0.0363$

The swallow tail behaviour indicates that a first order phase transition is taking place. The temperatures at and beyond the critical temperature do not exhibit such behaviour. The point where the two chemical potentials intersect is the point where the first order transition takes place.

We can see that the potential is decreasing in conjunction with the pressure. Along the same isotherm, a lower pressure will always have a lower potential, except for the swallowtail region. We ignore the swallowtail because it represents an unstable region that is not physical. The lower potential corresponding to a lower pressure means that the larger of the two black holes, once again, will be the favorable state.

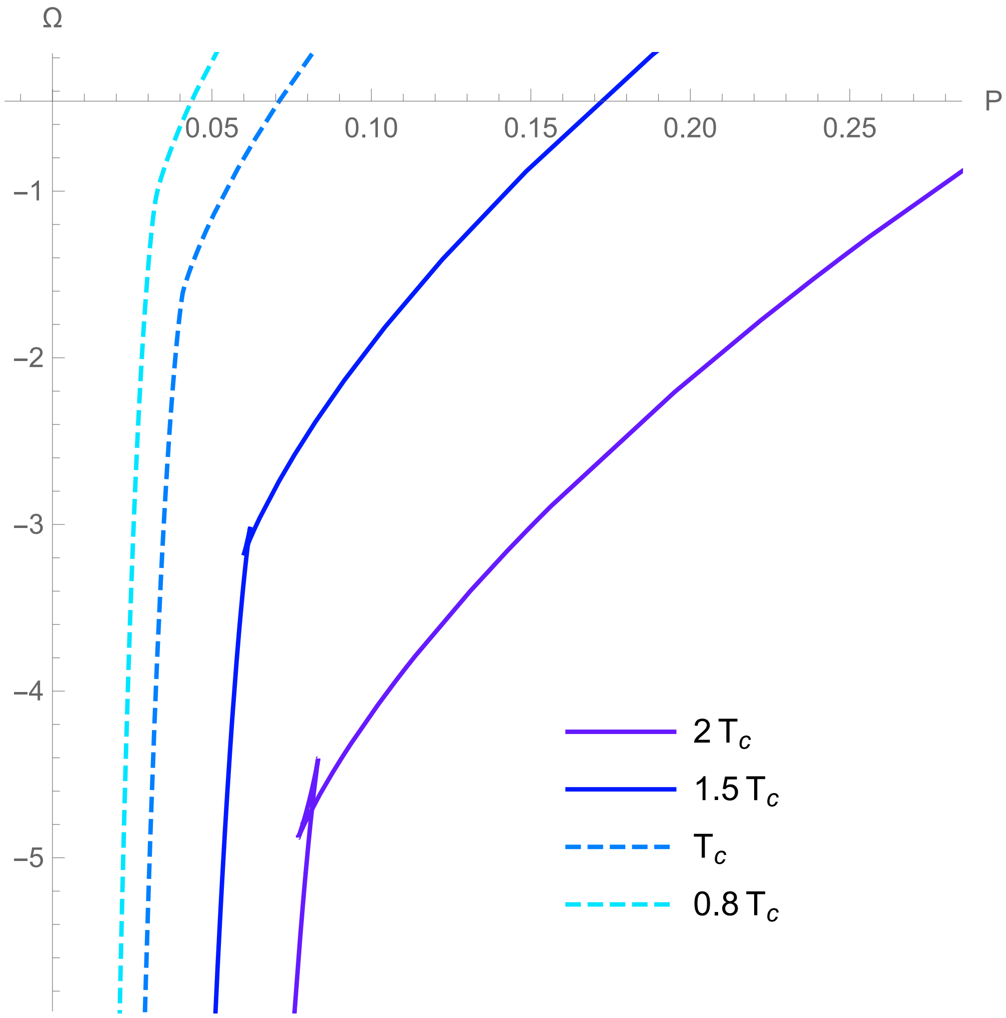

Figure 6.20: Isobaric plot for the temperature with respect to change in the horizon radius. $P = Pc$
$\Phi_e = 0.3$, $n = 1.98$, $p_h = 2$, $T_c = 0.338$

The way the pressure varies with the temperature does not differ at all, at the larger critical radius, from that of the other spacetimes or from the case where $\Phi_e^2 > 1$. However, we plot the variation of the potential with the pressure to determine which of the phases is favorable.

Expectedly, we find that the larger of both black holes is the favorable solution. As we stressed above, this is to be expected given that a minimisation of free energy will correspond to the maximisation of entropy. This is a good sign that our thermodynamic formulation is consistent.

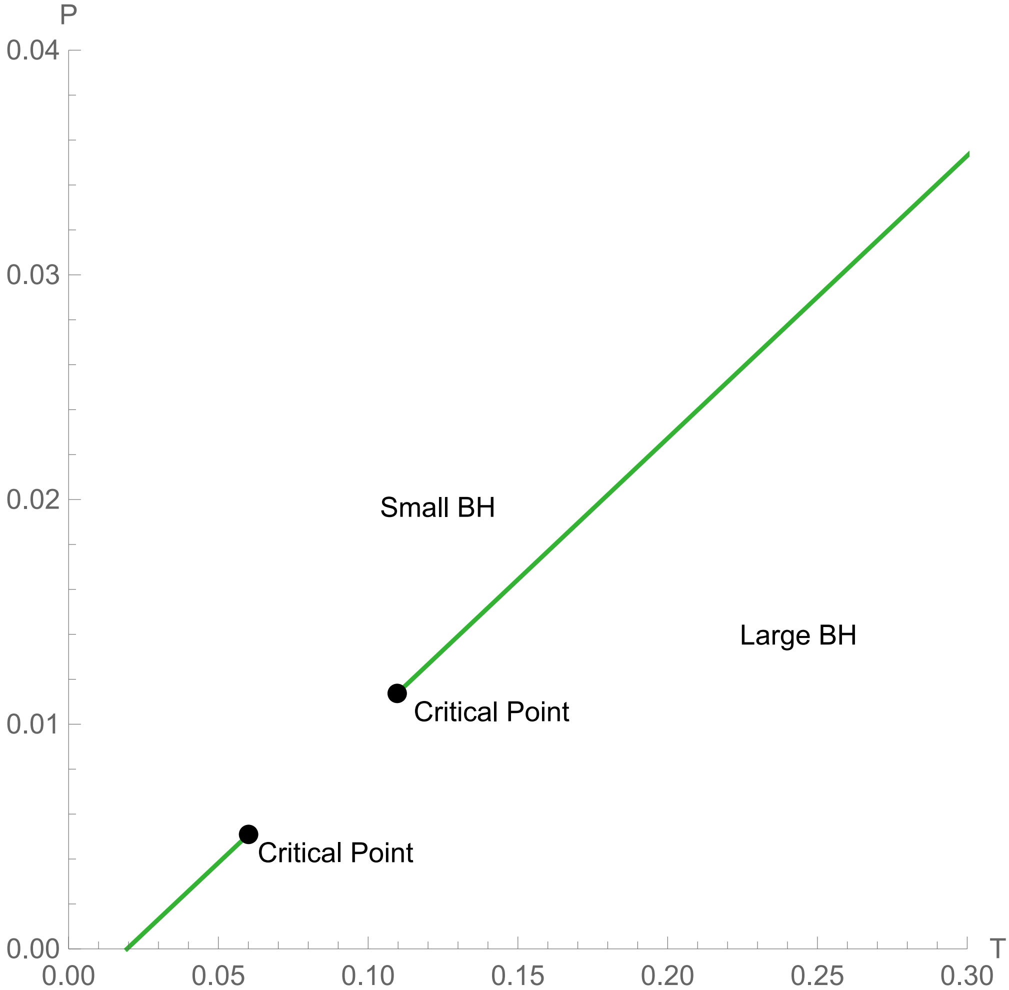

Figure 6.21: Isobaric plot for the temperature with respect to change in the horizon radius.
$\Phi_e = 0.5$, $n = 1.98$, $p_h = 2$, $P_{c_s} = 0.00510$, $T_{c_s} = 0.0601$, $P_{c_l} = 0.0114$, $T_{c_l} = 0.110$

The phase structure then differs from those discussed above because of the presence of the first order phase transition below the first critical point. The region in the middle has a second order phase transition where there is not distinction between the larger and smaller black holes. There is only one black hole.

Above the larger critical pressure and temperature a first order transition takes place between the smaller less favorable black hole and the more thermodynamically favourable large one. For the critical point that occurs at the lower critical pressure and temperature, the line separates the more stable phase at the larger horizon radius with the one at smaller horizon radius.

The phase structure can start to differ upon changing the parameters, however. This should come as no surprise when we recall that the main difference between the cases above and this case are the relative values of k and Φ_e^2. Upon tweaking the nut charge slightly we find

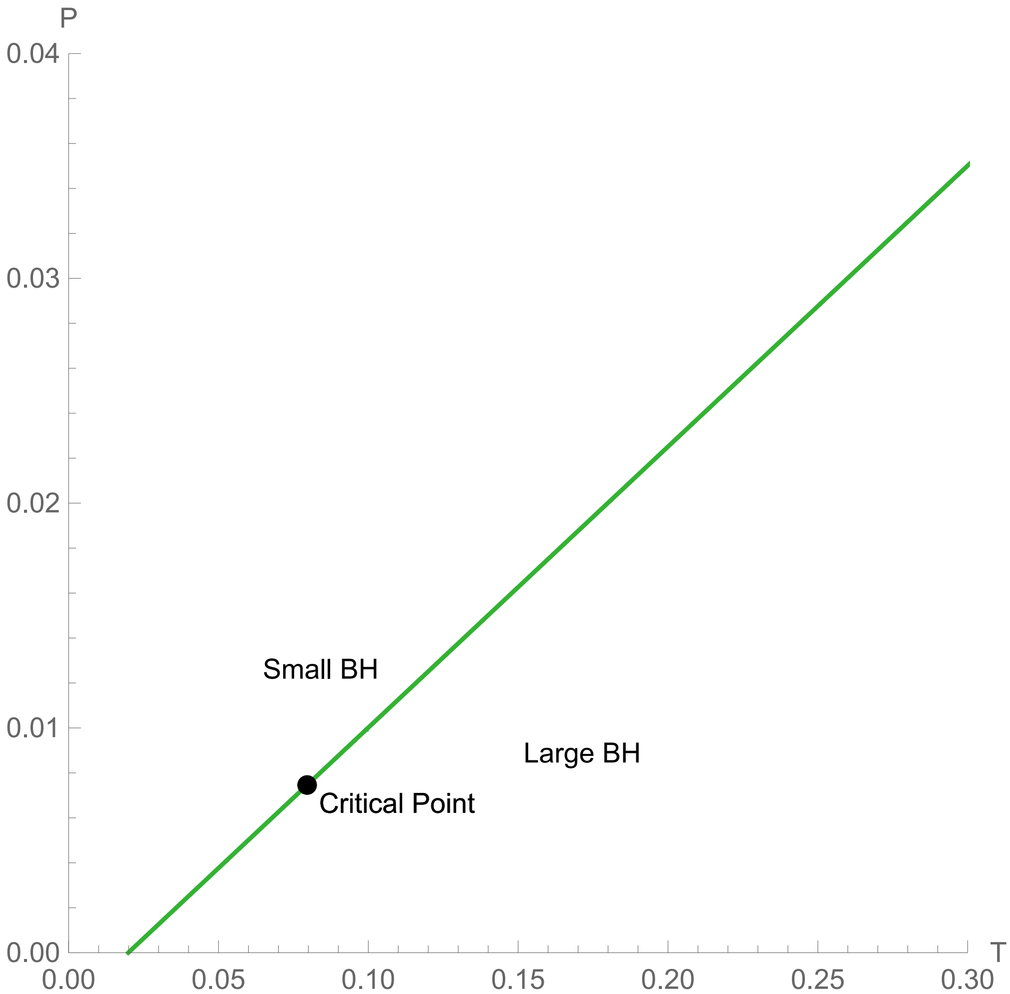

Figure 6.22: Isobaric plot for the temperature with respect to change in the horizon radius. $\Phi_e = 0.5$, $n = 2$, $p_h = 2$, $P_{c_s} = P_{c_l} = 0.00746$, $T_{c_s} = T_{c_l} = 0.0796$

Only one critical point exists in this case, as both of them overlap. This then entails that a first order transition occurs all along the $P - T$ plane, except for the critical point. At the critical point, there is no discontinuity in the entropy. However, a black hole crossing the line at any other point will experience a first order phase transition. The critical point vanishes completely when we increase n further.

Chapter 7

Conclusion

This chapter will address two topics. The first is presenting a summary to the results of this thesis and a discussion of the findings. In the second we will remark on possible avenues that could be explored, and ways in which this work could be improved.

In this thesis we endeavored to explore the phase structure presented by a family of solutions parameterised by the parameter k. The parameter k seperated between the different horizon geometries. For the hyperbolic case we had $k = -1$ while for the flat and the spherical cases, respectively, the value of k was 0 and 1.

We wanted to understand the influence the shape of the horizon has on the thermodynamics of the spacetime and the kind of phase structures that exist in each. The shape of the horizon, is indeed, found to play a significant part in the form and structure of the thermodynamic phases.

We started by formulating our thermodynamics and making sure of its consistency by checking that it satisfied several thermodynamic relations. This is very important since it allows us to explore the phase structure with the knowledge that our foundation is solid. We will briefly go through how we carried this out, and what the relations we satisfied were.

We started out by calculating the action first. We calculated the action through the use of the counterterm method. As opposed to the background subtraction method where you subtract a background spacetime from the action to cancel the divergences, the counterterm method uses curvature invariants on the boundary to cancel the divergences.

Aside from the action, it is very important to note the parameters that we fixed on our way to finding the action. These parameters are then our thermodynamic parameters. By fixing A_μ for example, we inherently fix both the magnetic charge and the electric potential.

In our case in particular, one of the more particular conserved quantities is the nut charge. We interpret it as some kind of gravito-magnetic charge. Its fixing comes from the integral of the two-form dual to that of the mass over the boundary. The nut charge itself differs from one horizon geometry to the next, but the fact that n itself is fixed at the boundary remains consistent.

Since we fixed A_μ at the horizon, the ensemble we used was a mixed one. This is because electric charges are not fixed, and can vary, while magnetic charges are fixed. We did not use it, but it is useful to note that there is a term that could be added to the action that fixes the electric charge at infinity that can take us to the canonical ensemble.

We made sure that when taking the partial derivatives of the our chemical potential with respect to the fixed parameters, we get their conjugate quantities. In a similar fashion, we made sure that the first law of thermodynamics holds. This is easy to verify through making sure the partial derivatives match those we expect to find.

The last two thermodynamic relations are the Smarr relation and the Gibbs-Duhem relation. The Smarr relation is concerned with the dimensions of the parameters used and is an application of Euler's theorem on quasi homogeneous functions. It assigns coefficients to each of the parameters depending on its dimension in length.

The Gibbs-Duhem relation is concerned with the relation between the chemical potentials in the system. It can be reached through assuming the quasi-homogeneity and taking the total differential. One part will traditionally be the internal energy, the rest describes a relation which is exactly the Gibbs-Duhem relation.

We then used forms in order to calculate the charges within the spacetime. We used that to justify the understanding of n as a gravito-magnetic charge. This is in direct analogy to the calculation of the electric and magnetic charges through the use of forms over the boundary.

Finally we started classifying the phases that exist and considering their stability. We presented the conditions for both thermal and mechanical stability before finding the stable phases for each geometry. We then examined the phase transitions and the critical behaviour within each.

The flat and hyperbolic geometries, both, had one critical point. This critical point came at a critical temperature and pressure above which a first order phase transition occurs. Below it, however, the transition is homogeneous and is of second order. We established the more preferable phase in both cases, it was found that the larger black hole is always favoured over the smaller one.

For the spherically horizon, we have more than one possibility. The distinction between both lies in the relation between Φ_e^2 and k. For $Phi_e^2 > k$, there is only one critical point, which we saw in the flat and hyperbolic cases. However, when $\Phi_e^2 < k$, we get two distinct critical points.

When this happens we see a different phase structure emerge that is unique to the spherical case alone. There are now two critical points between which a second order transition takes place. Below the smaller critical pressure and temperature, we have a first order phase transition. The same happens for pressures and temperatures greater than the larger critical values. It's important to note that the size of the gap depends on the charges present as was with the existence of the extra critical point.

The phase structure of all geometries is very interesting. In particular, the spherical case seems to have much more to give than it has currently let on. Studying the geometries in the same manner under traditional, as opposed to extended, thermodynamics could also prove to be very fruitful.

The relationship between the misner string and the charges is also worth examining further. In spherical case, the charges lie along the misner string.[37] However, this would not justify cases where the string does not exist such as the flat Taub-NUT-Ads metric. Even though there are no strings, there remains a disparity between the charges at the horizon and spatial infinity.

Appendix A

Komar Integrals

We will first discuss one-forms, hodge duals, and exterior derivatives before discussing conserved current. Following that we consider the divergence and Stokes' theorem. This paves the way to the calculation of conversed charges within a spatial hypersurface.

A.1 On Forms

No, not Plato's. Differential forms, or forms for short, are a special class of tensors. They are covariant tensors that are completely anti-symmetric. Since differential forms are tensors, their existence and characteristics are independent of the coordinate system in place.

For a spacetime of dimension n, the number of linearly independent p-forms are given by:

$$\frac{n!}{p!(n-p)!} \tag{A.1}$$

On a form we can define an exterior derivative on a one-form A denoted by dA. The exterior derivative can be taken on any form of size less than n to result in a form of dimension $n + 1$. We take the exterior derivative of a p-form in the following manner

$$dA_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \tag{A.2}$$

Where the square brackets on the indices represent an alternating sum over the indices within the brackets

$$
\begin{aligned}
A_{[ab]} &= \frac{1}{2!}\big(A_{ab} - A_{ba}\big) \\
A_{[abc]} &= \frac{1}{3!}\big(A_{abc} - A_{acb} + A_{cab} - A_{bac} + A_{bca} - A_{cba}\big)
\end{aligned}
\tag{A.3}
$$

A one-form on a manifold of dimension n using wedge notation can be written as:

$$
A = \sum_{i=1}^{n} A_{\mu_i} dx^{\mu_i} \tag{A.4}
$$

An n-form would take the form

$$
A = A\ dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_n} \tag{A.5}
$$

If we consider a form that exists on a particular hypersurface, let its dimension be of size p. On said hypersurface, we will have the equivalent of the n-form above but on the hypersurface. This will be a volume element of dimension p.

This brings us to one of the most important aspects in our use of forms. Let us take a one-form A. and take its exterior derivative.

$$
dA_{\mu\nu} = \frac{2}{2}\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big) = F_{\mu\nu} \tag{A.6}
$$

We then take the exterior derivative of the exterior derivative of A

$$
\begin{aligned}
d(dA_\mu) &= d(F_{\mu\nu}) \\
&= \frac{3}{3!}\big(\partial_\mu F_{\nu\rho} - \partial_\mu F_{\rho\nu} + \partial_\rho F_{\mu\nu} - \partial_\nu F_{\mu\rho} + \partial_\nu F_{\rho\mu} - \partial_\rho F_{\nu\mu}\big) \\
&= \frac{2\cdot 3}{3!}\big(\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu}\big) \\
&= \partial_\mu\big(\partial_\nu A_\rho - \partial_\rho A_\nu\big) + \partial_\rho\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big) + \partial_\nu\big(\partial_\rho A_\mu - \partial_\mu A_\rho\big) \\
&= \partial_\mu\partial_\nu A_\rho - \partial_\nu\partial_\mu A_\rho + \partial_\rho\partial_\mu A_\nu - \partial_\mu\partial_\rho A_\nu + \partial_\nu\partial_\rho A_\mu - \partial_\rho\partial_\nu A_\mu \\
&= 0
\end{aligned}
\tag{A.7}
$$

In moving from the second line to the third we used the fact that the two-form $F_{\mu\nu} = -F_{\nu\mu}$ is antisymmetric. In the line after we used the definition for $F_{\mu\nu}$ in terms of A_μ. In the last line we made use of the commutation of the partial derivative. Though our derivation above was made for the second exterior derivative of a one-form, this hold for forms.

This property becomes quite the useful one when we want to consider a charge or quantity within a spatial hypersurface.

We will now consider the hodge star operator. The hodge star is carried out on a differential form to give us it dual. It maps a p form on an n-dimensional manifold to an $(n-p)$ form. Let us consider an example below of the operator acting on an $n-p$ form to result in a p form.

$$\star A_{\mu_1 \dots \mu_{(n-p)}} = \frac{1}{p!} g_{\mu_1 \sigma_1} \dots g_{\mu_{(n-p)} \sigma_{(n-p)}} \epsilon^{\sigma_1 \dots \sigma_{(n-p)} \nu_1 \dots \nu_p} A_{\nu_1 \dots \nu_p} \tag{A.8}$$

Taking the second hodge star of the same form results in the form itself or its negative depending on the dimensionality, the signature of the manifold, and the number of indices of the form. If n is the dimension of the manifold, s its signature, and p the number of indices, operating with successive hodge stars will result in

$$\star \star A = (-1)^{s+p(n-p)} A \tag{A.9}$$

A.2 Generalised Stokes' Theorem

We will first discuss the divergence theorem, Stokes' theorem in three dimensional euclidean space before introducing Stokes' generalised theorem. The divergence theorem is arguably the most intuitive, Stokes' theorem in three dimensions is assumed to be familiar; both of which are special cases of the generalised theorem.

The divergence of a vector field in euclidean space is a scalar quantity that is calculated through taking the dot product of the gradient operator with the vector field. It can be interpreted as the outward *flux* from an infinitesimal point in space. For a vector field expressed in Cartesian coordinates $F(x, y, z)$ the divergence takes the following form.

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{A.10}$$

One widely used law that makes use of the divergence theorem is Gauss's law. The law linearly relates the electric flux through any closed surface to the amount of electrical charge inside the surface. For example, the net flux for through a surface containing charges that sum to zero is equal to zero.

We can now state the divergence theorem

$$\int\int\int dV\,(\nabla \cdot F) = \oint dA\,(F) \tag{A.11}$$

Let us use Gauss's law to make physical sense of the theorem. The theorem states that the flux that we can calculate by summing over all the sources within a volume V, is equal to the all the flux that will pass through a closed surface enclosing the boundary. To represent this we take the dot product of the outward pointing normal vector to the boundary with the field; this represents outgoing flux. So whenever we have sources within a surface, the net effect of the charges at the boundary is equal to the effect of the sources within the volume.

We can now move on to Stoke's theorem in three dimensions. Let us first consider the curl of a vector field, it represents the infinitesimal rotation about each point in space. The curl specifies, at each point, the axis of rotation for the field as well as its magnitude. A field whose curl is zero corresponds to a field that is irrational. Physically, this would correspond to a conservative field.

In euclidean space, the curl operates on a vector field through taking the cross product between the gradient operator, and the vector field. For a field $F(x,y,z)$ its curl takes the form

$$\nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)dx + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)dy + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_X}{\partial y}\right)dz \tag{A.12}$$

Stokes' theorem states that for any 2-dimensional surface in euclidean space, the integral of the curl of a field over a surface is equal to the integral of the field over a closed loop on the boundary of the surface. It's important to note that the direction taken along the boundary is non-trivial, it changes the sign of the integral. Convention is to take a path that moves clockwise about the outward pointing vector to the surface.

We will consider a simple example. If the surface were to be a disk, the curl would be integrated over the disk. This would then be equal to the integral of the field itself over the circle that encloses the disk. It is easy to see that upon flipping the unit normal to the disk we flip both the signs of the integral over the circle and the disk itself.

$$\int\int dA\,(\nabla \times F) = \oint dS\,(F) \tag{A.13}$$

In both the theorems above, we start with a some type of information about the change of the field within a volume or on a surface. Let us denote its dimension with n. We then relate this change in the field to the field itself at a boundary that has dimensions $n-1$. In going from $n = 1$ to $n = 0$ we would use the fundamental theorem of calculus; from a line to a difference between scalars.

To talk about the generalised Stokes' theorem we will invoke the language we used above in describing forms. We will simply state the theorem itself before moving on to its uses within general relativity.

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega \tag{A.14}$$

For a form ω with size p the exterior derivative $d\omega$ would have size $p+1$. If Ω represents a hypersurface, as it would in our cases of interest, then we can view $d\omega$ as the divergence of some field. The right side of the equation would then have us infer that ω is the field dotted into the normal unit vector to the boundary. The theorem holds under more rigorous scrutiny, but the present argument is sufficient for out case.

Let us consider for the example the one form ξ generated by the timelike killing vector ξ^{μ}. We first take its exterior derivative before operating on it with the hodge dual to get

$$\star d\xi \tag{A.15}$$

ξ is a one-form, upon taking the exterior derivative we get a two-form. It's hodge dual is then an $n-2$ form, where n is the dimension of the manifold. If we act with the exterior derivative again, we get an $n-1$ form

$$d \star d\xi \tag{A.16}$$

Taking the exterior derivative again will result in an n form. However the exterior derivative of an exterior derivative is zero. This allows us to write an integral over the manifold

$$\int_{M} d^2 \star d\xi = 0 \tag{A.17}$$

By Stokes' theorem we have

$$\int_{\partial M} d \star d\xi = 0 \tag{A.18}$$

We can separate the above boundary into two separate hypersurfaces in time which allows us to separate the integral over the boundary into two integrals

$$\int_{\Sigma_1} d \star d\xi - \int_{\Sigma_2} d \star d\xi = 0 \tag{A.19}$$

Taking both of these surfaces to be at spatial infinity, we can then use them to define a conserved quantity, in this case for a spacetime with no cosmological constant this corresponds to the mass

$$\int_{\Sigma_1} d \star d\xi = M \tag{A.20}$$

Then by applying Stokes' theorem we reach

$$\int_{\partial\Sigma_1} \star d\xi = M \tag{A.21}$$

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