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The American University in Cairo
School of Sciences and Engineering

# Study of the Thermodynamics of Charged Rotating Black Holes in Five-Dimensional Anti-de Sitter Spacetimes

*A Thesis Presented in Partial Fulfillment of the Requirements for the Degree of Master of Science in Physics*

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# Acknowledgements

For me to finish this thesis would not have been possible without many people to whom I will be eternally grateful.

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Hassan ElSayed,
Cairo, Egypt
3 June 2020

# Preface

This thesis is the culmination of the past two years of my life which I have dedicated, for the most part, to studying the general theory of relativity and doing research in black hole thermodynamics. While I have worked on two other small research projects in GR during that time, I consider the research presented here as the most valuable of this period.

One of the most challenging parts of conducting my first research project in theoretical physics was finding good resources to learn about certain topics that I needed to understand. Many of the references I had to go through were – as you can imagine – quite challenging to follow. Nevertheless, I have come across several theses that were extremely helpful and aided me in understanding certain topics that I was trying to learn. Some were useful in explaining certain concepts and others were helpful in explaining the detailed procedure of certain calculations.

My aim in this thesis is to simultaneously present to a defense committee my own research along with my understanding of the theoretical concepts that underlie it, and to offer as clear and detailed an explanation as possible of the topics that I discuss. In doing the latter I hope that maybe someday I could benefit someone in the way that many others have benefited myself. With that in mind, I have made a few decisions which I hope will improve the readability of this thesis. First, I have tried to do two things that can be at odd with each other: give a complete discussion of the elements that I am presenting, while keeping each discussion very compact. In order to do this I have moved many parts of the discussion to appendices. This helps make the discussions of the core chapters concise, while also presenting the information in the appendices in stand-alone discussions for anyone interested only in reading these parts.

I have also attempted to give detailed proofs and derivations in all my discussions. Some of the proofs that I found for certain formulae were too complicated and I have attempted to come up with shorter proofs myself to present them here. Because reading a long proof can make the reader feel disconnected from the big picture, many proofs are placed in individual subsections. The uninterested reader can simply jump to the end of any proof (denoted by the conventional end-of-proof symbol "□").

Aside from proofs, I have tried, to the best of my judgement, to present the details of every relevant calculation in ways to help anyone who is trying to reproduce these calculations. Here I include both previously-available calculations as well as my own calculations in Chapter 4.

# Conventions and Notations

### Units

We work in the Plank natural units system. In this system we have:

- Speed of light $c = 1$,
- Gravitational constant $G = 1$,
- Reduced Planck constant $\hbar = 1$,
- The Boltzmann constant $k_{\mathrm{B}} = 1$,
- The Coulomb constant $\frac{1}{4\pi\varepsilon_0} = 1$.

### Relativity and Tensors

Unless otherwise stated, all metrics are assumed to have Lorentzian signatures $(-, +, +, ...+)$. All mathematical results will be expressed in light of this assumption.

We mostly use the *abstract index* notation of Roger Penrose wherein indices denoted by Latin letters $\{a, b, c...g\}$ indicate a tensors type rather than its components in a particular basis. Nevertheless, Greek letters are occasionally used to emphasize the fact that we are referring to a specific basis. On a $D$-dimensional manifold, Greek indices run from 0 to $D - 1$. The specific Latin letters $\{i, j, k\}$ denote only spatial components in a particular basis and hence run from 1 to $D - 1$.

The Einstein summation convention is used by default whenever an index of any of the above types is raised an lowered.

When the coordinates are given by $x^a = (x^0, ...x^{D-1})$, the partial derivative operator notation that follows is

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

We also sometimes do deal with spatial vectors in 3 dimensions. These vectors are denoted by upper arrows,

$$\vec{x} = (x^1, x^2, x^3).$$

## General Symbols in Mathematics

| | |
|---|---|
| $\forall$ | For all |
| $\exists$ | There exists |
| $\square$ | End of proof |
| $\mid$ | Such that |
| $:=$ | Equal by definition |
| $\dagger$ | Complex conjugate |

## Sets and Topology

| | |
|---|---|
| $\emptyset$ | Empty set |
| $\in$ | $x \in A$, $x$ is an element of $A$ |
| $\cup$ | $A \cup B$, union of $A$ and $B$ |
| $\cap$ | $A \cap B$, intersection of $A$ and $B$ |
| $\subset$ | $A \subset B$, $A$ is contained in $B$ |
| $\backslash$ | $A\backslash B$, $B$ subtracted from $A$ |
| $\bar{A}$ | Closure of $A$ |
| $\dot{A}$ | Boundary of $A$ |
| $f : \mathbb{A} \to \mathbb{B}$ | $f$ maps $x \in \mathbb{A}$ to an element $f(x) \in \mathbb{B}$ |
| $\bigcup\limits_{t\in S}$ | $\bigcup\limits_{t\in S} A_t$, the union of all elements $A_t$ with $t$ taking all values in $S$ |

## Differential Geometry

| | |
|---|---|
| $\nabla$ | Covariant derivative |
| $\Box \equiv \nabla^a \nabla_a$ | D'Almbert operator |
| $\mathrm{d}$ | Exterior derivative |
| $\wedge$ | Exterior/wedge product |
| $\star$ | Hodge star operator |
| $()$ | $T_{(ab)}$, symmetrization of $T_{ab}$ |
| $[]$ | $T_{[ab]}$, anti-symmetrization of $T_{ab}$ |
| $\mathcal{L}_V$ | Lie derivative along the vector field $V$ |
| $R_{abcd}$ | Riemann curvature tensor |
| $R_{ab}$ | Ricci tensor |
| $R$ | Ricci scalar |
| $C_{abcd}$ | Weyl tensor |

## Causal Structure

| | |
|---|---|
| $J^+(S)$ | Causal future of $S$, definition 1.7 |
| $I^+(S)$ | Chronological future of $S$, definition 1.8 |
| $\mathscr{I}^\pm$ | Future/past null infinity, definition 1.9 |
| $D^\pm(S)$ | Future/past domain of dependence, definition B.4 |

## Symbols Commonly Used in Text

| | |
|---|---|
| $\mathcal{M}$ | Manifold |
| $T_p(\mathcal{M})$ | Tangent space at a point $p \in \mathcal{M}$ |
| $\partial\mathcal{M}$ | Spatial boundary of $\mathcal{M}$ at infinity |
| $\mathcal{B}$ | Black hole region |
| $\mathcal{H}$ | Black hole event horizon hypersurface |
| $\mathcal{S}^n$ | $n$-sphere |
| $g_{ab}$ | Metric tensor |
| $g$ | Determinant of $g_{ab}$ |
| $h_{ab}$ | Induced metric tensor at infinity |
| $\sigma_{ab}$ | Foliation metric at fixed time and radial coordinates |
| $n^a$ | Spacelike outward-pointing unit normal vector |
| $u^a$ | Timelike future-pointing normal vector |
| $T_{ab}$ | Energy-momentum tensor |
| $K_{ab}$ | Extrinsic curvature tensor |
| $\delta_{ab}$ | Kronecker delta |
| $\epsilon$ | Levi-Civita tensor |
| $\tilde{\epsilon}$ | Levi-Civita symbol |

# Introduction

Despite the success of General Relativity in the past century, a quantum description of gravity remains one of the most active areas of research in theoretical physics today. While gravity becomes dominant at large scales where classical physics is an appropriate effective theory, the study of very massive systems on a very small scale would require a quantum theory of gravity. A quantum description of gravity is also needed to have a grand unified theory of all the fundamental forces. The most prominent candidate for a theory of quantum gravity is currently String Theory [1].

The discovery of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence in [2] shows a duality between gravity solutions from String Theory in anti-de Sitter (AdS) spacetime and conformal field theories (CFT) that reside on its boundary. The announcement of this conjecture was an important step for String Theory as it presents a connection between the latter and Quantum Field Theory, while also enabling a non-perturbative approach to Quantum Field Theory calculations with some boundary conditions. Particularly, in a certain limit which we will discuss, the duality means that the study of certain classes of Quantum Field Theory in flat spacetime can be mapped to classical black hole solutions in five-dimensional anti-de Sitter spacetimes, the latter being considered an approximation to solutions from String Theory. A strong interest in the study of five-dimensional black hole solutions in anti-de Sitter spacetimes then follows naturally from the AdS/CFT correspondence, even though these solutions lack strong astrophysical or cosmological interest. In fact, the study of black hole solutions in five-dimensional AdS spacetimes reveals important information about strongly coupled SU($N$) gauge theories.

Conformal field theories are quantum field theories that are invariant under conformal transformations. The latter are transformations that preserve the angles but not necessarily the lengths [3]. Renormalization of these theories can break this conformal symmetry, leading to a quantum anomaly known as *conformal anomaly*. This anomaly is characterized by the value of the trace of the energy momentum tensor. One of the predictions of the AdS/CFT correspondence is a relation between the latter and the energy-momentum tensor of the gravitational theory. Using this, the conformal anomaly of the boundary CFT can be computed from the gravitational theory.

Another interesting relation between the two theories is an equivalence between the AdS background energy of black hole solutions and the vacuum energy of the dual CFT on the boundary.

Black holes are curious objects: they are regions which no object can escape from, regardless of the object's speed. This type of behavior only comes from the general relativistic description of gravity and cannot be predicted from classical physics. While in classical physics we can have massive bodies with gravitational fields strong enough that their escape velocities are larger than the speed of light, this is different from a body curving spacetime in a way that does not allow for any timelike or null objects to escape from it. Moreover, black holes can become massive and complex bodies, with many other particles and formations lying beyond their event horizons. Nevertheless, these presumably complex objects often can be completely described by the simple laws of thermodynamics, so long as we make the appropriate attributions between the quantities in the first law and the black hole parameters.

The study of black hole thermodynamics in anti-de Sitter spacetimes presents certain challenges that do not exist in flat spacetimes. Particularly, certain isometric charges as well as the action become divergent on the boundary. While a number of regularization schemes exist [4, 5], none of them is the sole "canonical" choice that everyone agrees on. Particularly popular solutions are the "background subtraction method" and the "counterterms subtraction method". The latter has several advantages which we will discuss in the thesis. Furthermore, different expressions exist in the literature for some thermodynamical quantities like the energy and angular velocity in higher-dimensions AdS spacetimes [6]. The choice between these expressions has a direct effect on the validity of the first law of black hole thermodynamics, which becomes more elusive in five dimensional AdS spacetimes than it is in four dimensions. Different solutions have been proposed for this issue, for example in [6], [7] and [8].

A black hole in five dimensions can have up to two independent angular momenta. It can also have an electric or a magnetic charge. Our present goal is to study a general solution of electrically charged black holes in five-dimensional AdS space with two different rotation parameters. Simpler cases like neutral black holes with only one non-vanishing rotation parameter [9] and black holes with two non-vanishing and different rotation parameters [9, 7] have been studied extensively.

The metric under consideration here was presented in [10] where the energy of the black hole was calculated by integration of the first law of thermodynamics. This is based on arguments presented in [6] which claim that other ways of calculating the energy in five-dimensional and higher anti-de Sitter spacetimes will violate the first law. However, we find this method of deriving an expression for the mass to be particularly unfavorable since it does not relate the definition of the energy to a Killing vector and hence to an isometry. By assuming the validity of the first law and getting a mass by integrating its right-hand side we also can no longer check if the first law is actually valid for a particular solution.

Finally, this method for calculating the energy does not lead to any data which can be used to verify the duality between the gravitational theory in the bulk and a conformal field theory residing on the conformal boundary.

Conversely, it was argued in [7] that the first law does in fact hold if we calculate the energy using the counterterms subtraction method. However, the first law in this case, as suggested in [7], needs to be altered to account for variations in the boundary metric. The general form of the first law was derived in [7] for an action which did not include a Chern-Simons term, and was applied to a neutral rotating black hole. It will be interesting to check if this claim holds for the case of the solution in [10], which was derived from an action that contains Maxwell and Chern-Simons terms.

When presenting our work in Chapter 4, we will start by checking the validity of a set of thermodynamical relations for the solution in [10]. We will then derive a regularized expression for the action using the counterterms subtraction technique. From this action, one can derive the quasi-local energy-momentum tensor, and from that, a regularized expression for the energy of the black hole as well. With this counterterms energy, one can then check the validity of the modified first law of black hole thermodynamics that was proposed by Skenderis and Papadimitriou. We expect that it will be possible to satisfy this relation for the Einstein-Maxwell-Chern-Simons case that we are interested in.

Furthermore, we will calculate the conformal anomaly and Casimir energy of the bulk gravitational action as well as using the conformal field theory on the boundary and compare the results. We expect of course that the AdS/CFT conjecture will imply the equivalence of both sets of calculations in the aforementioned limit.

# Chapter 1

# Black Holes

## 1.1 Preliminaries

Before presenting the core topic of the chapter we will set some ground rules for the rest of the thesis. Aside from certain basic information, the thesis is self contained. Furthermore, many elementary concepts (e.g. Killing vector fields and hypersurfaces) are thoroughly discussed where they are needed. Additionally, Appendix A gives a review of the mathematical operations from Differential Geometry that will be used in the thesis. In general it might be a good idea that the reader starts by reading Appendices A-C before going to the core chapters since the latter make many references to equations in those appendices.

Throughout the thesis we will be using the Levi-Civita connection and will denote the covariant derivative by $\nabla$. This means that the connection coefficients $\Gamma^a_{bc}$ are always the Christoffel coefficients given by

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left( \partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab} \right), \tag{1.1}$$

where $g_{ab}$ is the metric tensor of the manifold. Specifying the Levi-Civita connection directly implies metric compatibility,

$$\nabla_c g_{ab} = 0, \tag{1.2}$$

and torsion-freedom,

$$\Gamma^c_{[ab]} = 0. \tag{1.3}$$

The ordering of the Riemann tensor indices follows [11], so that the Riemann (1,3) tensor is given by

$$R^d{}_{cab} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^d_{al}\Gamma^l_{bc} - \Gamma^d_{bl}\Gamma^l_{ac}. \tag{1.4}$$

It follows that the Ricci tensor is defined by

$$R_{ab} = R^l{}_{alb}, \tag{1.5}$$

and the Weyl tensor by

$$\begin{aligned} C_{dcab} =& R_{dcab} - \frac{2}{D-2}\left(g_{d[a}R_{b]c} - g_{c[a}R_{b]d}\right) \\ &+ \frac{2}{(D-1)(D-2)} g_{d[a}g_{b]c}R, \end{aligned} \tag{1.6}$$

where $D$ is the dimension of the manifold, and $R$ is the Ricci scalar,

$$R = R^a{}_a. \tag{1.7}$$

Finally, the Einstein tensor is defined as

$$G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R. \tag{1.8}$$

Following these notations, we have the **first Bianchi identity** given by

$$\nabla_e R_{dcab} + \nabla_b R_{dcea} + \nabla_a R_{dcbe} = 0. \tag{1.9}$$

**Proof:** Let us consider the Riemann tensor in local inertial coordinates. In these coordinates the Christoffel symbols of course vanish (although their derivatives do not necessarily vanish). Then using equation (1.1) in (1.4) gives

$$
\begin{aligned}
R_{dcab} \mathrel{\hat{=}} g_{dl}\left(\partial_a \Gamma^l_{bc} - \partial_b \Gamma^l_{ac}\right) \\
&= \frac{1}{2} g_{dl} g^{lm} \big(\partial_a \partial_b g_{cm} + \partial_a \partial_c g_{mb} - \partial_a \partial_m g_{bc} - \partial_b \partial_a g_{cm} \\
&\qquad - \partial_b \partial_c g_{ma} + \partial_b \partial_m g_{ac}\big) \\
&= \frac{1}{2}\left(\partial_a \partial_c g_{db} - \partial_a \partial_d g_{bc} - \partial_b \partial_c g_{da} + \partial_b \partial_d g_{ac}\right). \qquad (1.10)
\end{aligned}
$$

The symbol "$\hat{=}$" signifies "equal in a particular coordinate system". Taking the covariant derivative,

$$
\begin{aligned}
\nabla_e R_{dcab} &\mathrel{\hat{=}} \partial_e R_{dcab} \\
&= \frac{1}{2}\left(\partial_e \partial_c \partial_a g_{bd} + \partial_e \partial_d \partial_b g_{ca} - \partial_e \partial_c \partial_b g_{da} - \partial_e \partial_d \partial_a g_{cb}\right). \qquad (1.11)
\end{aligned}
$$

Taking permutations of the above equation leads to

$$
\nabla_b R_{dcea} \mathrel{\hat{=}} \frac{1}{2}\left(\partial_b \partial_c \partial_e g_{ad} + \partial_b \partial_d \partial_a g_{ce} - \partial_b \partial_c \partial_a g_{de} - \partial_b \partial_d \partial_e g_{ca}\right), \qquad (1.12)
$$

and

$$
\nabla_a R_{dcbe} \mathrel{\hat{=}} \frac{1}{2}\left[\partial_a \partial_c \partial_b g_{ed} + \partial_a \partial_d \partial_e g_{cb} - \partial_a \partial_c \partial_e g_{db} - \partial_a \partial_d \partial_b g_{ce}\right). \qquad (1.13)
$$

Given that regular partial derivatives commute, it is easy to see that the sum of the previous three equations cancels out,

$$
\nabla_e R_{dcab} + \nabla_b R_{dcea} + \nabla_a R_{dcbe} = 0. \qquad (1.14)
$$

Since this is a tensorial equation, it is true in any coordinate frame.

□

We also give the **second Bianchi identity**:

$$
\nabla^a R_{ab} = \frac{1}{2} \nabla_b R. \qquad (1.15)
$$

**Proof:** Contracting both sides of the first Bianchi identity:

$$
\begin{aligned}
0 &= g^{ce} g^{da} \left( \nabla_b R_{dcae} + \nabla_e R_{dcba} + \nabla_a R_{dceb} \right) \\
&= g^{ce} (\nabla_l R^a{}_{cae} - \nabla_e R^a{}_{cal} + \nabla_a R^a{}_{ceb}) \\
&= g^{ce} (\nabla_b R_{ce} - \nabla_e R_{cb} - \nabla_a R_c{}^a{}_{eb}) \\
&= \nabla_l R^e{}_e - \nabla_e R^e{}_b - \nabla_a R^{ea}{}_{eb} = 0 \\
&= \nabla_b R - \nabla_e R^e{}_b - \nabla_a R^a{}_b.
\end{aligned} \tag{1.16}
$$

Changing the dummy index $e$ to $a$ in the second term allows us to recast this as

$$
0 = \nabla_b R - 2\nabla_a R^a{}_b, \tag{1.17}
$$

which gives us the desired results

$$
\nabla^a R_{ab} = \frac{1}{2} \nabla_b R. \tag{1.18}
$$

□

## 1.2 Schwarzschild Solution

### 1.2.1 The Schwarzschild Metric

The Schwarzschild metric is a (relatively) simple solution to the Einstein equation. It will serve as a nice introduction to the concept of black holes as well as playground for showing certain properties related to them (for instance, why they are "black").

What makes the Schwarzschild solution a simple one compared to other solutions in four-dimensional spacetime, is the fact that it is static and spherically symmetric. We proceed to define what these terms mean. Let us start by defining *stationary* spacetimes.

**Definition 1.1.** A (four-dimensional) spacetime is said to be **stationary** if it admits a time-oriented Killing vector[1] $\chi = \partial/\partial t$.

Recall from §C.3 that the existence of such a Killing vector implies that the spacetime is invariant under time translation. This gives sense to the definition above. Of course this is not always the case. For instance, an expanding universe does not have the aforementioned

[1] See §C.3

timelike Killing vector. It is hence not stationary in the sense that it does not look the same at every instance in time; it grows in size.

A rotating spacetime is stationary if it has a constant angular velocity vector at each point in space and time. This obviously means that the spacetime looks the same at every moment in time; it is therefore time-independent. A non-rotating stationary spacetime is *static*.

**Definition 1.2.** A spacetime is said to be **static** if it admits a timelike Killing vector field that is orthogonal to a hypersurface of that spacetime [12].

It is easy to that if the spacetime is rotating, then the timelike Killing vector field will not be perpendicular to spatial hypersurfaces. Note also that definition 1.2 by its own implies that the spacetime is stationary.

**Definition 1.3.** A spacetime is said to be **spherically symmetric** if it is invariant under the isometries[2] group SO(3)[3] operating on the two-sphere[4]

$$t = \text{constant}, \; r = \text{constant}.$$

Without any further suspense, let us announce the Schwarzschild metric.

$$ds^2 = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \tag{1.19}$$

where $\mathrm{d}\Omega^2$ is the metric on the unit two-sphere and is given by

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \tag{1.20}$$

The metric being static and spherically symmetric has an isometry group $\mathbb{R} \times \mathrm{SO}(3)$. Evidently, the first factor represents the symmetry with respect to time-translation, and (as we mentioned in §C.1.2) the second factor represents invariance under rotation in $\mathbb{R}^3$. Note that if $m = 0$ then the metric reduces to that of the Minkowski spacetime. If $m \neq 0$ but $r \gg |m|$, the metric is almost Minkowskian. We call this *asymptotically flat*. We give proper definition of this term below. The definition uses several terms which were defined defined in Appendix B:

**Definition 1.4.** A hypersurface $\Sigma$ with induced metric $h_{ab}$ and extrinsic curvature $K_{ab}$ is called **asymptotically flat** if:
1. There exists a smooth invertible and bijective map between $\Sigma$ and $\mathbb{R}^3 \backslash B_R$, where $B_R$ is

[2] See §C.3.
[3] See §C.1.2.
[4] See §B.1

a closed ball centered on the origin in $\mathbb{R}^3$.
2. If $(x^i)_i$ are the cartesian coordinates on $\Sigma$ then $h_{ij} = \delta_{ij} + \mathcal{O}(x^{-1})$ and $K_{ij} = \mathcal{O}(x^{-2})$ as $x := \sqrt{x^i x_i} \to \infty$. $\delta_{ij}$ is the Kroncker-Delta.

It is worth noting that we know from *Birkhoff's theorem* that the Schwarzschild metric is the unique spherically symmetric solution to the Einstein field equation in vacuum[5][11].

Having presented the final solution, let us see how it can actually be derived from the Einstein field equations. The simplest spherically symmetric spacetime we know is the Minkowski metric

$$ds^2_{\text{Minkowski}} = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2. \tag{1.21}$$

We will generalize this to a more general metric where each term is multiplied by a function of spacetime which is yet to be determined. In a mathematical language we call these functions, which we assume to exist and give the sought-after solution, *ansatz functions.* In order to maintain spherical symmetry, the ansatz function multiplied by $\mathrm{d}\theta^2$ and $\sin^2\theta \mathrm{d}\phi^2$ must be the same. Furthermore, all ansatz should not depend on either $\theta$ or $\phi$. An educated guess, which will turn out to be helpful, is to write the ansatz functions in terms of exponentials,

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + e^{2\gamma(r)} r^2 \mathrm{d}\Omega^2. \tag{1.22}$$

It can also be shown that, without loss of generality, the function $\gamma(r)$ can be taken equal to 0.

**Proof:** We are free to make a change of coordinate

$$r \to \tilde{r} e^{\gamma}, \tag{1.23}$$

with corresponding one-form

$$\begin{aligned} \mathrm{d}\tilde{r} &= e^{\gamma}\mathrm{d}r + e^{\gamma} r \mathrm{d}\gamma \\ &= \left(1 + r\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right) e^{\gamma}\mathrm{d}r. \end{aligned} \tag{1.24}$$

The metric can then be expressed in terms of the new coordinate as

[5] To be more accurate, the Schwarzschild solution is the unique solution *outside* the black hole region. We will shortly discuss what we mean by "black hole" and "black hole region".

$$ds^2 = -e^{2\alpha}\mathrm{d}t^2 + \left(1 + r\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right)^{-2} e^{2(\beta-\gamma)}\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2. \tag{1.25}$$

We could now define new functions $\tilde{\alpha}(\tilde{r})$ and $\tilde{\beta}(\tilde{r})$ such that

$$\begin{aligned}\tilde{\alpha}(\tilde{r}) &= \alpha(r),\\ \tilde{\beta}(\tilde{r}) &= \beta(r) - \gamma(r).\end{aligned} \tag{1.26}$$

This allows us to re-write the metric as

$$ds^2 = -e^{2\tilde{\alpha}(\tilde{r})}\mathrm{d}t^2 + e^{2\tilde{\beta}(\tilde{r})}\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2. \tag{1.27}$$

Forgetting our original no-tilde functions and coordinate $r$, we can drop the tildes from the notation of the new ansatz and coordinate, and re-write the metric as

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{1.28}$$

□

To solve the Einstein field equation, we start by giving the values of all independent and non-null Christoffel symbols:

$$\begin{aligned}&\Gamma^t_{tr} = \partial_r\alpha, \quad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha, \quad \Gamma^r_{rr} = \partial_r\beta,\\ &\Gamma^r_{\theta\theta} = -re^{-2\beta}, \quad \Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta, \quad \Gamma^\theta_{r\theta} = r^{-1},\\ &\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{r\phi} = r^{-1}, \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}.\end{aligned} \tag{1.29}$$

The components of the Ricci tensor can then be found via a direct but tedious calculation. Luckily we can use the Maple software to automatically get those:

$$\begin{aligned}R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + (2/r)\partial_r\alpha\right],\\ R_{rr} &= -(\partial_r\alpha)^2 - \partial_r^2\alpha + \partial_r\alpha\partial_r\beta + (2/r)\partial_r\beta,\\ R_{\theta\theta} &= 1 - e^{-2\beta}\left[r(\partial_r\alpha - \partial_r\beta)\right],\\ R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}.\end{aligned}$$

The Ricci scalar is given by

$$R = -2e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2(\partial_r\alpha - \partial_r\beta)}{r} + \frac{1-e^{2\beta}}{r^2}\right]. \tag{1.30}$$

The Einstein equation in vacuum comes down to equating the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R,$$

to 0. Of course each component of $G_{ab}$ cancels by itself. We can also add the $tt$ and $rr$ components and multiply by $e^{-2\beta}$, leading to

$$\begin{aligned} 0 &= e^{2(\beta-\alpha)}R_{tt} + R_{rr} \\ &= \frac{2}{r}(\partial_r\alpha + \partial_r\beta). \end{aligned} \tag{1.31}$$

This implies that

$$\alpha(r) = -\beta(r) + c, \tag{1.32}$$

with $c$ some constant. We can easily set $c$ equal to 0 by rescaling our time coordinate as

$$t \to te^{-c},$$

leading to

$$\alpha(r) = -\beta(r). \tag{1.33}$$

Using the $G_{\theta\theta}$ component, we can arrive at

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1, \tag{1.34}$$

which implies that

$$\partial_r(re^{2\alpha}) = 1, \tag{1.35}$$

allowing us to write

$$e^{2\alpha} = 1 - \frac{C}{r}, \tag{1.36}$$

where $C$ is some constant. The Schwarzschild metric is then given by

$$ds^2 = -\left(1 - \frac{C}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{C}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{1.37}$$

Identifying $C$ with $2m$ we recover the metric in (1.19).

We express the constant $C$ as $2m$ because of convenience since it will turn out to be proportional to the black hole mass, as we will see later in this chapter. In the same manner, in the charged black hole solution we express another constant as $q$ because it will turn out to be proportional to the black hole charge. It is very important to note that $m$ and $q$ are constants that we get by solving the Einstein field equations; their relations to the mass and charge of the black hole respectively are not explicit. So whenever we have parameters in a solution labeled $m$ and $q$ (or $M$ and $Q$), we can not simply assume that these are exactly the mass and charge of the black hole respectively. In fact, we will see that the relation between the black hole mass and $m$, and that between the black hole charge and $q$ will change depending on the dimension and asymptotic topology of the spacetime.[6]

To make this distinction clear and to not confuse symbols, throughout this thesis we will denote all metric constants that come out of solving the Einstein equation by small letters, and the black hole conserved quantities like mass and charge by capital letters.

### 1.2.2 The Schwarzschild Black Hole

The metric in equation (1.19) posses two singularities, one for $r = 2m$ and another for $r = 0$. The second turns out to be a *curvature singularity* and cannot be eliminated by a change of coordinates. Strictly speaking, this means that the point $r = 0$ does not belong to the manifold. The first singularity is in fact what we call a *coordinates singularity*; it can be eliminated by a change of coordinates. To see this, we first define the *tortoise coordinate*

$$r^* = r + 2m \ln\left|\frac{r}{2m} - 1\right|. \tag{1.38}$$

The *Eddington-Finkelstein coordinates* are then found by simply replacing $t$ by

[6]The *asymptotic topology* of the spacetime is the topology of the spacetime manifold when the radial coordinate goes to infinity.

$$v = t + r^* \tag{1.39}$$

The metric in the new Eddington-Finkelstein coordinates is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2 \tag{1.40}$$

We see that the point $r = 2m$ is no longer a singularity. The point $r = 0$ remains however a singularity as we have claimed.

We now wish to look at the light cone in these coordinates. First note that, since the solution is spherically symmetric, we can always fix $\theta$ and $\phi$ without loss of generality. If we do this, our metric simply becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r \tag{1.41}$$

The conditions for null curves are

$$\begin{aligned} &\frac{\mathrm{d}v}{\mathrm{d}r} = 0, \\ &\frac{\mathrm{d}v}{\mathrm{d}r} = 2\left(1 - \frac{2m}{r}\right)^{-1}. \end{aligned} \tag{1.42}$$

Figure 1.1 shows the evolution of the light cones for the Schwarzschild solution as a function of the position $r$. Here we see a striking result: future-directed causal curves can only move further away from the point $r = 2m$. To put it differently, nothing inside the surface $r = 2m$ can escape outside of it. When we say "nothing" we obviously mean both objects (which form timelike curves) and light (which forms null curves). The surface $r = 2m$ is called the *event horizon.* Since light cannot escape the even horizon, we cannot see anything inside it and the object appears to be black. That is why we refer to these objects as *black holes.*

## 1.3 Event Horizons

### 1.3.1 The Causal Structure

Having laid out the foundations for the concept of black holes, we now wish to formulate a definition for the black hole region. First let us start by recalling some important definitions which will be used in this as well as other sections in the thesis.

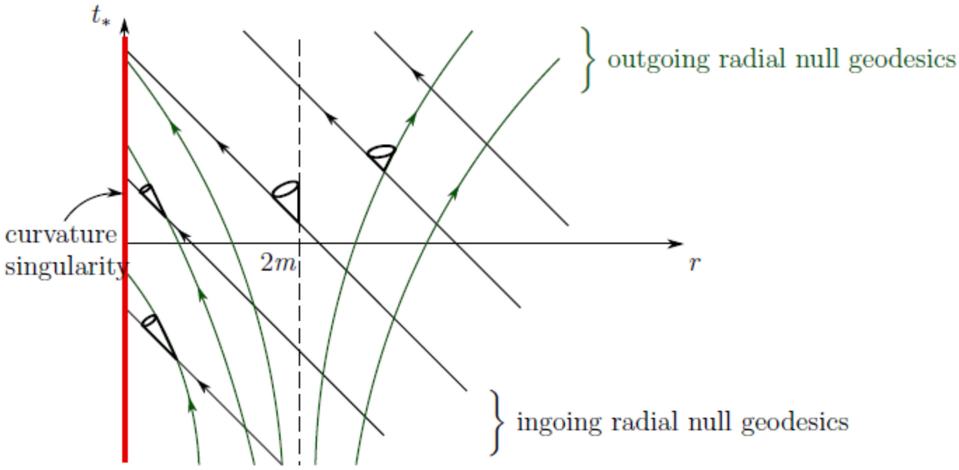


Figure 1.1: The tiling of the light-cones before, at and after the event horizon at $r = 2m$. *Credit: [13]*

**Definition 1.5.** Let $E$ and $F$ be two normed vector spaces [7], and $U \subset E$, $V \subset F$ two open sets. A map $f : U \to V$ is called a **diffeomorphism** if and only if $f$ is differentiable on $U$, bijective, and $f^{-1}$ is differentiable on $V$ [14].

In this thesis, we are specifically interested in diffeomorphisms between manifolds. Now, recall that a *causal curve* is simply a curve whose tangent vector to the spacetime manifold is everywhere timelike or null.

**Definition 1.6.** A spacetime manifold $\mathcal{M}$ is **time-orientable** if it admits a smooth, nowhere vanishing timelike vector field $\hat{t}^a$. A causal vector $x^a$ is **future-directed** if $x^a \hat{t}_a < 0$. A causal curve is a **future-directed causal curve** on $\mathcal{M}$ if its tangent vector is future-directed everywhere on $\mathcal{M}$ [15].

**Definition 1.7.** Given any subset $S$ of a spacetime $(\mathcal{M}, g)$, the **causal future** of $S$, denoted $J^+(S)$, is the set of points that can be reached from $S$ by following a future-directed causal curve [11].

**Definition 1.8.** The **chronological future** of $S$, denoted $I^+(S)$, is the set of points that can be reached from $S$ by following a future-directed timelike curve [11].

Analogous definitions hold for the causal and chronological pasts $J^-(S)$ and $I^-(S)$ [11].

**Definition 1.9.** The **future** (respectively **past**) **null infinity**, denoted $\mathscr{I}^+$ (respectively $\mathscr{I}^-$) is the set of points which are approached asymptotically by future-directed (respectively past-directed) null rays that can escape to infinity [11].

[7] On which a norm is defined.

To define the black hole region we will need to use the hypersurface of $I^+$, which is not actually contained in the physical spacetime $(\mathcal{M}, g)$. We therefore define the "unphysical" manifold $(\tilde{\mathcal{M}}, \tilde{g})$ which, amongst other properties, contains $J^{\pm}(\mathscr{I}^{\mp})$. Whenever we mention $J^{\pm}(\mathscr{I}^{\mp})$ we mean this set as defined in $(\tilde{\mathcal{M}}, \tilde{g})$.

**Definition 1.10.** Let $(\mathcal{M}, g)$ be a spacetime. The **black hole region** is defined as [16]

$$\mathcal{B} := \mathcal{M} \backslash [\mathcal{M} \cap J^-(\mathscr{I}^+)]. \tag{1.43}$$

### 1.3.2 Event Horizons

Following on from the previous definition, we can easily find a definition for event horizons [16].

**Definition 1.11.** The **event horizon** of a black hole region is the boundary of the black hole region

$$\mathcal{H} := \dot{\mathcal{B}}. \tag{1.44}$$

In the original Schwarzschild metric there was a coordinate singularity at the horizon as we have seen; the component $g_{rr}$ goes to infinity. Conversely, the component $g^{rr}$ given by

$$g^{rr} = 1 - \frac{2m}{r}, \tag{1.45}$$

goes to 0 at the event horizon. Beyond the event horizon it is easy to see that this component changes sign, and hence the the radial coordinate $r$ changes signature and becomes timelike. In such convenient coordinate systems then, the location of the event horizon can be found by equating $g^{rr}$ to 0 and solving for $r$. Furthermore, you can easily see that, since the $g_{tt}$ component is simply $-1/g_{rr}$, the time coordinate too changes sign beyond the horizon and becomes spacelike. Subsequently, in the Schwarzschild solution, the timelike Killing vector $\partial_t$ also changes sign and becomes spacelike beyond the event horizon [11].

We have cheated a little whenever we used the term "event horizon." The truth is we mean the "future event horizon" as there is another event horizon called (as you might have guessed) the "past event horizon." The latter is not really related to black holes but rather to another phenomenon called *white holes*. We are not going to discuss white holes here since they do not help much in our ultimate goal of tackling black hole thermodynamics. In conclusion, whenever we say "event horizon" we could be a little more rigorous and say "future event horizon." This "abbreviation" however remains widely used in the literature.

### 1.3.3 Killing Horizons

We now look at a related concept, that of Killing horizons.

**Definition 1.12.** Let $\Sigma$ be some hypersurface embedded in a manifold $\mathcal{M}$, and $\xi^a(x)$ a Killing vector in $\mathcal{M}$. $\Sigma$ is a **Killing horizon** if $\forall\, x \in \Sigma, \xi^a(x)\xi_a(x) = 0$.

Generally speaking, there is no connection between Killing horizons and event horizons. But in spacetimes with time-translation symmetry they are related. Carter has shown that, for static black holes, the event horizon is a Killing horizon for the Killing vector $\chi^a = (\partial_t)^a$ [11]. If the spacetime is stationary with some rotational Killing vector $\eta^a$, Hawking has shown that, under certain energy conditions, the event horizon must be a Killing horizon for some vector field. Often, the latter is a linear combination of $\chi^a$ and $\eta^a = (\partial_\phi)^a$. This linear combination depends on the angular velocity of the horizon, $\Omega_H$, and is given by ([11])

$$\xi = \partial_t + \Omega_H \partial_\phi. \tag{1.46}$$

We now suppose that we have an event horizon $\mathcal{H}$ that is also a Killing horizon with respect to a Killing vector field $\xi^a$. From Definition 1.12, it follows that the gradient of $\xi^a\xi_a$ is normal to $\mathcal{H}$. Since $\xi^a$ is also normal to $\mathcal{H}$, then there exists a function $\kappa$ on $\mathcal{H}$ such that

$$\nabla^a(\xi^b\xi_b)\Big|_{\mathcal{H}} = -2\kappa\xi^a \tag{1.47}$$

The main reason for our interest in $\kappa$ is that it will turn out to be related to the Hawking temperature of the black hole (see §2.3). The name "surface gravity", however, comes from a classical interpretation of this quantity, particularly in static spacetimes. Using the fact that $\xi^a$ is normal on the horizon and Killing's equation, it is easy to find a useful formula for $\kappa$,

$$\kappa^2 = -\frac{1}{2}(\nabla^a\xi^b)(\nabla_b\xi_a)\Big|_{\mathcal{H}}. \tag{1.48}$$

Now, consider a test particle orbiting some gravitational source. This particle experiences a force proportional to its acceleration. Recall that, for a normalized velocity vector, the four-acceleration $a^a = U^b\nabla_b U^a$ is related to the redshift factor $V$ at infinity by

$$a_a = \nabla_a \ln(V) \tag{1.49}$$

The magnitude of this acceleration is therefore [8]

$$a = \sqrt{a^a a_a} = V^{-1}\sqrt{\nabla_a V \nabla^a V} \tag{1.50}$$

[8] The author admits that this choice combination of variables and indices is not very well chosen.

Hence, $V$ represents the "redshifted" acceleration of a test particle at the surface of the black hole as seen by an observer at infinity. This is why we call it the surface gravity. You might feel that the fact that this seemingly arbitrary quantity is assigned a name – particularly one which only makes sense in the specific case of asymptotically flat, static spacetimes – does not seem to be very well motivated. But this quantity will turn out to be extremely important because of its aforementioned relation to the black hole temperature. Therefore, it is worth giving a specific name since it is constantly referred to in the literature.

## 1.4 Charged, Rotating Black Holes

We now direct our attention to charged, rotating black holes. To find a charged black hole solution we can proceed similarly to the way we found the Schwarzschild metric, only now we are not looking for a vacuum solution of the Einstein equation, but rather one for which there is a non-null electromagnetic energy-momentum tensor,

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}. \tag{1.51}$$

Another addition to the recipe is that the equations of motion are Einstein's *and* Maxwell's equations [11]. Both previous points mean that we also have ansatz functions as the components of the electromagnetic four-potential $A$.

A *rotating* black hole solution on the other hand is rather tricky to derive, and it took physicists and mathematicians a few decades to find one. The difficulty stems from the fact that a rotating black hole (or any rotating object for that matter) does not have spherical symmetry but only axial symmetry [11].

In four-dimensional asymptotically flat spacetime, the general charged, rotating black hole solution is called the Kerr-Newman solution. It is given by

$$\begin{aligned} ds^2 = &-\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right)\rho^2 + (\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2\frac{\Delta}{\rho^2} \\ &- \left((r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\right)^2\frac{\sin^2\theta}{\rho} \end{aligned} \tag{1.52}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{1.53}$$

$$\Delta = r^2 - 2mr + a^2 + q^2. \tag{1.54}$$

In this coordinate system the black hole is rotating in the $\phi$-direction. The parameters $m$, $a$ and $q$ will turn out to be the mass, angular momentum per unit mass and electric charge of the black hole, respectively. We will discuss the generic procedure for calculating such quantities in §1.5. For now, we just want to note that we can have a solution for a rotating, non-charged black hole simply by setting $q = 0$. This is called the **Kerr metric**. We can also have a solution for a non-rotating, charged black hole by setting $a = 0$. This is called the **Reissner-Nordström** metric.

Let us look at the Kerr black hole. First, notice that this is not a static solution to the Einstein field equation: the black hole is rotating. The event horizon will then not be the Killing horizon of the timelike Killing vector $\partial_t$. To find the event horizon we can equate $g^{rr}$ to 0. This comes down to solving

$$\Delta|_{\mathcal{H}} = (r^2 - 2mr + a^2)|_{\mathcal{H}} = 0, \tag{1.55}$$

since $g^{rr} = \Delta/\rho$. The discriminant of the above quadratic equation can be easily found to be

$$\mathrm{Disc}_m(\Delta) = m \pm \sqrt{m^2 - a^2}. \tag{1.56}$$

We have three possibilities when looking for solutions: $m < a$, $a = m$ and $m > a$. The first possibility presents no real solutions. The second possibility $m = a$ is an unstable solution since adding any randomly small amount of matter will turn it to the third case. The third case $m > a$ has two solutions,

$$r_\pm = m \pm \sqrt{m^2 - a^2}, \tag{1.57}$$

which correspond to an inner event horizon (at $r_-$) and an outer event horizon (at $r_+$). As an outside observer, only the latter seems relevant. As we mentioned, neither of these radii will be the null surface of the timelike Killing vector $\chi = \partial_t$. The norm of this vector field is found to be

$$\chi^a \chi_a = -\frac{1}{\rho^2}\left(\Delta - a^2 \sin^2\theta\right). \tag{1.58}$$

At $r = r_+$ this norm is positive,

$$\chi^a \chi_a = \frac{a^2 \sin^2\theta}{\rho^2} \geq 0. \tag{1.59}$$

Hence, the timelike Killing vector switches signature even before the event horizon, except at the two points $(r_+, 0)$ and $(r_+, \pi)$ where the term $\sin^2\theta$ obviously vanishes [11].

The surface on which the timelike Killing vector becomes null is called the **stationary limit surface**. The region between it and the outer event horizon is called the **ergosphere**. Figure 1.2 shows a depiction of this structure. Inside the ergosphere the timelike Killing vector changes sign but you can still escape from it before hitting the event horizon [11]. Evidently, the ergosphere is an interesting region and we will talk more about it in the next chapter.

In the following section we will discuss how we can calculate the angular momentum of the Kerr black hole, along with other conserved black hole quantities. But for now, we simply would like to see how we can define the black hole's angular *velocity*. This of course can be done for the complete Kerr-Newman solution but we will only do it for the Kerr metric for simplicity. Imagine a photon emitted in the $\phi$-direction at some radius $r$ and on the plane $\theta = \pi/2$. Since the trajectory is null, we have

$$\begin{aligned} ds^2 &= g_{tt}\mathrm{d}t^2 + g_{t\phi}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) + g_{\phi\phi}\mathrm{d}\phi^2 \\ &= 0, \end{aligned} \tag{1.60}$$

which leads to

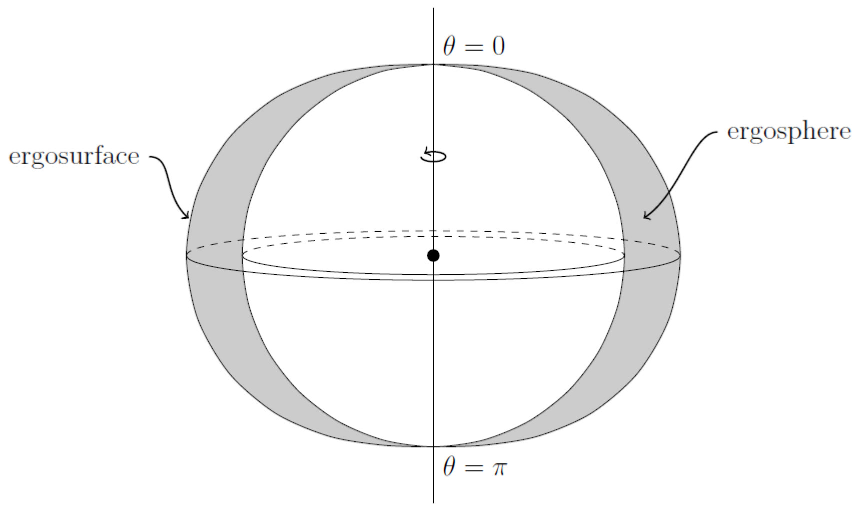


Figure 1.2: The ergosphere is the grey region between the stationary limit surface – where the timelike Killing vector becomes null – and the black hole's event horizon. *Credit: Ref. [15]*

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \tag{1.61}$$

We can define the angular velocity as the minimum solution to the above equation at the outer horizon:

$$\Omega_H := \left.\frac{\mathrm{d}\phi}{\mathrm{d}t}\right|_{r_+} = \frac{a}{a^2 + r_+^2}. \tag{1.62}$$

Another interesting place to evaluate the expression in (1.61) is at the spacetime boundary, and in some cases we define the "angular velocity of the black hole" as the one at the outer horizon minus that at the boundary,

$$\Omega := \left.\frac{\mathrm{d}\phi}{\mathrm{d}t}\right|_{r_+} - \left.\frac{\mathrm{d}\phi}{\mathrm{d}t}\right|_{r\to\infty} \tag{1.63}$$

However, in our case, both equations (1.62) and (1.63) are equivalent because the last term in (1.63) vanishes.

The last relevant quantity that we will discuss in this section is the electric potential $\Phi$. Like the temperature and the angular velocity, this quantity will depend on the radial coordinate $r$ in general, and by convention we define it at the outer horizon. The derivation of a formula for the electric potential turns out to be a very laborious task which was accomplished by Carter [17]. It was done by requiring that the black hole be in equilibrium (i.e. vanishing Lorentz force) and that the electric potential be constant on the horizon (at least for axisymmetric black holes). It is given by ([7])

$$\Phi_H = -\xi^a A_a|_{\mathcal{H}},$$

where $\xi^a$ is the horizon Killing vector. This potential can present divergences for certain black hole solutions. Fortunately, since this is a *potential*, we can define it with respect to a reference that cancels these divergences. We therefore define the potential with respect to the boundary at infinity by ([18])

$$\begin{aligned}\Phi &= \Phi_H - \Phi(r)|_{r\to\infty} \\ &= -\xi^a A_a|_{\mathcal{H}} + \xi^a A_a|_{r\to\infty}\end{aligned} \tag{1.64}$$

## 1.5 Conserved Charges

As we discuss in Appendix C, conserved quantities play a major role in any physical theory. We have discussed several aspects of black holes, but so far we have not discuss any conserved quantities related to them. In this section we will present how conserved quantities of black holes – namely the electric charge, mass and angular momentum – can be defined and calculated. These quantities gain extra importance from what is called the "no-hair theorem", originally stated by Werner Israel in 1967. The theorem says that black holes can be completely characterized by these parameters: mass, charge and angular momentum. Furthermore, we will see in the next chapter that the thermodynamics of black holes can be completely described by these quantities along with the quantities described in the previous sections (i.e. the even horizon, angular velocity and electric potential).

### 1.5.1 Electric Charge

Appendix C discusses how charges are defined and calculated in curved spacetimes. We have already seen how the classical Coulomb charge can be calculated using this recipe. Let us see how this can be applied to a charged black hole. Since we do not care about rotation at this point, we will use the Reissner-Nordström solution, which can be recovered by setting $a = 0$ in the Kerr-Newman metric (1.52), giving

$$ds^2 = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\phi\right), \tag{1.65}$$

where

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \tag{1.66}$$

Recall that the electromagnetic current is given by

$$\nabla_b F^{ab} = J^a. \tag{1.67}$$

Using the result in (C.34) with the above current, we have

$$\begin{aligned} Q &= \int_\Sigma \mathrm{d}^3x\sqrt{-h}n_a J^a \\ &= \int_\Sigma \mathrm{d}^3x\sqrt{-h}\nabla_b F^{ab}. \end{aligned} \tag{1.68}$$

Using Stokes' theorem (particularly result (C.31)), we have

$$\begin{aligned} Q &= \int_{\partial\Sigma} \sqrt{-\gamma} n_a u_b F^{ab} \\ &= \int_{\partial\Sigma} \sqrt{-\gamma} F^{rt} \\ &= \int_{S^2} \mathrm{d}\theta\, \mathrm{d}\phi\, r^2 \sin^2\theta \frac{q}{4\pi r^2} \\ &= q. \end{aligned} \tag{1.69}$$

As before, $\gamma_{ab}$ is the induced metric on $\partial\Sigma$ and $u^a$ is the outward-pointing normal vector to $\partial\Sigma$. You can see that this is a virtually effortless calculation thanks to the effort that we have made in defining the procedure in §C.2.

### 1.5.2 Komar Integrals

#### Mass

The next (and perhaps even first) obvious quantity to look at is the black hole mass (or energy). The concept of black hole mass is rather tricky, and there are many ways to define what is the "mass" of the black hole. The mass calculation is actually an important part of the original work done in this thesis so we will spend some time to discuss different aspects of this concept.

Several parts in this section rely heavily on results developed in Appendix C. It is recommended that the reader reviews this Appendix before proceeding with this section.

From our classical mechanics intuition, the energy is a conserved quantity which is related to time-translation symmetry. We can always find a time-translation Killing vector $\chi^a$ for stationary solutions, and we have seen in §C.3 that we can construct a conserved current from this Killing vector,

$$J^a_T = \chi_b T^{ab}. \tag{1.70}$$

From the conservation of $T^{ab}$ and Killing's equation, we can easily show that this current is divergenceless (see §C.3 for detailed calculations). Therefore, we can find a conserved quantity by integrating it over a spacelike hypersurface $\Sigma$,

$$E_T = \int_\Sigma \mathrm{d}^{(D-1)}x \sqrt{-h}\, n_a J^a_T, \tag{1.71}$$

where $D$ is the spacetime dimension and $h$ the determinant of the induced metric on $\Sigma$.

There is, however, a number of problems with this definition. The first one is that this energy goes to zero for vacuum solutions, where $T^{ab} = 0$. You can easily check that this is the case for example in the Schwarzschild solution. But there are several reasons that make us want to define a non-zero energy. For example, the black hole may form as the result of the collapse of a star. The matter of the star is now contained beyond the horizon and so, outside the black hole's horizon, $T^{ab} = 0$. However, it would still make sense to have a value for the mass since 1) we would like the original energy of the star to be conserved, and 2) the black hole still provides a gravitational field which should have some sort of energy associated with it.

We now consider a new current

$$J_K = \chi_b R^{ab}. \tag{1.72}$$

Its divergence is given by

$$\nabla_a J_K^a = (\nabla_a \chi_b) R^{ab} + \chi_b (\nabla_a R^{ab}). \tag{1.73}$$

We use the fact that $(\nabla_a \chi_b)$ is antisymmetric while $R^{ab}$ is symmetric to eliminate the first term on the right-hand side. We recall that

$$\nabla_a R^{ab} = \frac{1}{2} \nabla^b R,$$

which leads to

$$\nabla_a J_K^a = \frac{1}{2} \chi_a \nabla^a R = 0. \tag{1.74}$$

The last equality is due to the fact that the directional derivative of $R$ vanishes along a Killing vector as we have seen in §C.3. We also know from §C.1.1 that this implies that

$$\mathrm{d} \star J_K = 0, \tag{1.75}$$

and that we can associate a conserved charge with this this current,

$$M = \frac{1}{4\pi} \int_\Sigma \star J_K. \tag{1.76}$$

The $1/4\pi$ factor is a normalization factor, added for convenience. We may now chose this charge to be our definition for the black hole mass. The mass in equation (1.76) is called the **Komar mass**.

To get a sense of this formula, let us turn our attention to the Schwarzschild solution. In a move that feels like we are diverting from the main topic, let us recall some results from Newtonian mechanics. The reasons for this will become obvious shortly. The acceleration of a unit-mass body in a gravitational potential $\Phi\left(\vec{r}\right)$ is given by

$$\vec{a} = -\vec{\nabla}\Phi(\vec{r}). \tag{1.77}$$

The Poisson differential equation for Newtonian gravity is [11]

$$\nabla^2\Phi = 4\pi\rho \tag{1.78}$$

where $\rho$ is the matter density. We would like to see how the Schwarzschild solution compares to this picture in a "Newtonian limit". We follow the definition of [11] for Newtonian limit as: (1) particles are moving with respect to the speed of light, (2) the gravitational field is weak, and (3) the field is static. In relativistic theories, a slowly moving particle means that

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \tag{1.79}$$

so the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 \cong 0. \tag{1.80}$$

The weak field condition means that the metric $g_{ab}$ can be perturbed around the flat-space metric $\eta_{ab}$,

$$g_{ab} = \eta_{ab} + h_{ab}, \qquad |h_{ab}| \ll 1. \tag{1.81}$$

The static field condition simply means that $\partial_0 g_{\mu\nu} = 0$, which leads to

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}. \tag{1.82}$$

To first order in $h_{\mu\nu}$ this leads to

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}. \tag{1.83}$$

Plugging that into the geodesic equation, we have

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2. \tag{1.84}$$

The $\mu = 0$ component is

$$\left(\frac{dx^0}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 = 0, \tag{1.85}$$

since $\partial_0 h_{00} = 0$. This means that $dt/d\tau$ is constant. In non-relativistic mechanics this is of course true since time is regarded as an absolute quantity which is measured the same in all reference frames. This is a good check that we are on the right track. The spatial components of (1.84) are

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}. \tag{1.86}$$

If we compare (1.86) to (1.77), we identify

$$h_{00} = -2\Phi, \tag{1.87}$$

leading to

$$g_{00} = -(1 + 2\Phi). \tag{1.88}$$

In the Schwarzschild solution this leads to the usual Newtonian potential

$$\Phi = -\frac{m}{r}, \tag{1.89}$$

where we recall that Newton's constant in normal units is set to 1. This allows us to identify the *geometric parameter* $m$ in the Schwarzschild solution with the mass in the Newtonian limit.

We now calculate the mass of the Schwarzschild black hole using the Komar integral. Expressing the Komar mass in the Schwarzschild coordinates, we get

$$M = \frac{1}{4\pi} \int_{\partial\Sigma = \mathcal{S}^2} \mathrm{d}^2 x \sqrt{-\gamma}\, u_a\, n_b\, \nabla^a \chi^b. \tag{1.90}$$

The one-forms associated with the normal unit vectors are

$$\begin{aligned} u_a &= \left(-\sqrt{-g_{tt}} \;\; 0 \;\; 0 \;\; 0\right), \\ n_a &= \left(0 \;\; \sqrt{g_{rr}} \;\; 0 \;\; 0\right). \end{aligned} \tag{1.91}$$

Given these forms, the only surviving components of the term $u_a n_b \nabla^a \chi^b$ is the $u_0 n_1 \nabla^0 \chi^0$ term. It can be easily seen that it gives

$$u_0 n_1 \nabla^0 \chi^0 = -\nabla^0 \chi^1 = -\frac{m}{r^2}. \tag{1.92}$$

Finally, we have

$$M = \frac{1}{4\pi} \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \; r^2 \sin\theta \left(\frac{m}{r^2}\right) \tag{1.93}$$

$$= m, \tag{1.94}$$

which is the result that we were hoping to get. When developing a new theory, it is always important to show that it reduces to the results of previous well-established theories under the correct assumptions or approximations. Luckily, this turns out to be the case here: the Komar mass yields the same result as Newtonian mechanics in the Newtonian limit.

### Angular Momentum and Komar Integrals

Our procedure for arriving at (1.76) did not rely on the fact that $\chi^a$ is the stationary Killing vector. In fact, we can generate conserved currents for any Killing vector $\xi^a$ in our spacetime. It is therefore useful to have a formula for conserved charges, like the one in equation (1.76), which is expressed in terms of a generic Killing vector $\xi^a$.

We define the generic current

$$J^a[\xi] := \xi_b R^{ab}, \tag{1.95}$$

to which is associated a conserved charge

$$Q[\xi] \propto \int_\Sigma \star J[\xi]. \tag{1.96}$$

We would then like to express $\star J[\xi]$ solely in terms of $\xi$. In §C.3 we showed that

$$\nabla_a \nabla_c K^a = R_{cb} K^b. \tag{C.65}$$

This allows us to write the current as

$$J[\xi] = \nabla_b \nabla_a \xi^b. \tag{1.97}$$

Multiplying by the right-hand side by $g^{ab} g_{bc} = \mathbb{1}$ gives

$$\begin{aligned} J &= \nabla^b (\nabla_b \xi_c) \\ &= 2\nabla_b (\mathrm{d}\xi)_{bc}. \end{aligned} \tag{1.98}$$

Using equation (A.13) with $A_{bc} = (\mathrm{d}\xi)_{bc}$, it is easy to re-write this as

$$\begin{aligned} J &= -(-1)^{2(4-2)} \times 2 \star \mathrm{d} \star \mathrm{d}\xi \\ &= -2 \star \mathrm{d} \star \mathrm{d}\xi. \end{aligned} \tag{1.99}$$

Then, using (A.6), we have the equality

$$\begin{aligned} \star J &= -1 \times -1 \times (-1)^{3(4-1)} \times 2\mathrm{d} \star \mathrm{d}\xi \\ &= -2\mathrm{d} \star \mathrm{d}\xi. \end{aligned} \tag{1.100}$$

Identifying the above with equation (1.96), we have

$$Q[\xi] \propto \int_\Sigma \mathrm{d} \star \mathrm{d}\xi. \tag{1.101}$$

Finally, using Stokes' theorem, we have the important result

$$Q[\xi] \propto \int_{\partial\Sigma} \star \mathrm{d}\xi. \tag{1.102}$$

This formula is called the **Komar integral**. The reason we have used the proportionality sign instead of an equality followed by a normalization factor like the previous section is because we use different normalization factors depending on the spacetime dimensions. For the case four dimensions case, the formula for the mass of the black hole is found by re-instating the $1/4\pi$ factor and dividing by the $-2$ factor in equation (1.100),

$$M = -\frac{1}{8\pi} \int_{\partial\Sigma} \star \mathrm{d}\chi. \tag{1.103}$$

The expression for the angular momentum is found by using the Killing vector that corresponds to rotational invariance. For the metric in (1.52) the rotation invariance is seen by noting that the metric is independent of $\phi$ (while no longer independent of $\theta$ in this non-spherically symmetric case), and the corresponding Killing vector is thus $\eta := \partial_\phi$. The Komar integral for the angular momentum in this case is

$$J_\phi = -\frac{1}{8\pi} \int_{\partial\Sigma} \star \mathrm{d}\eta. \tag{1.104}$$

In the presence of a non-vanishing cosmological constant, it was discussed in [19] that these formulae can be generalized (in four dimensions) to include an extra term as follows:

$$Q[\xi] = -\frac{1}{8\pi} \int_{\partial\Sigma} \star \mathrm{d}\xi - \frac{\Lambda}{4\pi} \int_{\Sigma} \xi \tag{1.105}$$

### 1.5.3 Brown-York Quasi-local Charges

By now it should be clear that the definition of energy in General Relativity is not unique. We have already seen two different ways of defining the energy of a black hole and have expressed a general preference for one of them (the so-called Komar energy) out of pure convenience. But nothing stops us from finding other definitions for the energy. Whether or not a new definition is more convenient than the previous one is of course a different story.

In 1992 David Brown and James York presented an expression for calculating the energy [20] which is based on the Hamilton-Jacobi formalism. The resulting definition of energy will be used in our calculations in Chapter 4.

Before proceeding, we should note that this section makes reference to several concepts developed in §B.3. It also makes reference to the equations for the gravitational action and energy-momentum tensor which were developed in Appendix D.

**Motivation for the Idea**

The principal idea is to consider a classical mechanical system with configuration space $\mathbb{Q}$, and Lagrangian $L : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$, meaning the Lagrangian is assumed to be first-order in the canonical variables, and may depend explicitly on time. We consider initial and final configurations $(q_1^a, t_1)$ and $(q_2^a, t_2)$. The action is thus written as

$$S^1[q^a(t)] = \int_{t_1}^{t_2} L(q^a, \dot{q}^a, t)\mathrm{d}t. \tag{1.106}$$

If we now vary the endpoint to $(q_2^a + \delta q, t_2 + \delta t)$, the variation of the action can be written as the sum of two parts, the first due to the variation of $q$ which we denote $\delta_q$ and the second due to the variation of $t$ which we denote $\delta_t$.

$$\delta S^1 = \frac{\partial S^1}{\partial q}\delta q + \frac{\partial S^1}{\partial \dot{q}}\delta \dot{q} + \frac{\partial S^1}{\partial t}\delta t, \tag{1.107}$$

where the suppressed indices of the vectors $q$ and $\dot{q}$ are implied. The first term gives

$$\frac{\partial S^1}{\partial q}\delta q = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q \mathrm{d}t. \tag{1.108}$$

The second term gives

$$\begin{aligned}\frac{\partial S^1}{\partial \dot{q}}\delta \dot{q} &= \int \frac{\partial S}{\partial \dot{q}}\delta \frac{\mathrm{d}}{\mathrm{d}t} q \mathrm{d}t \\ &= \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} - \int \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\delta q \mathrm{d}t.\end{aligned} \tag{1.109}$$

To get to the last line we have used integration by parts. The third terms can be written as

$$\begin{aligned}\frac{\partial S^1}{\partial t}\delta t &= \int_{t_1}^{t_2} L\mathrm{d}t\delta t\\ &= L\delta t|_{t_1}^{t_2}\\ &= (p^a\dot{q}_a - E)\,\delta t|_{t_1}^{t_2}.\end{aligned} \tag{1.110}$$

The combination of the three terms leads to

$$\delta S^1 = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)(\delta q + \dot{q}\delta t) + \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} + (E+L)\delta t|_{t_1}^{t_2} \tag{1.111}$$

$$= (\text{terms giving the equations of motion}) + \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} + (E+L)\delta t|_{t_1}^{t_2}. \tag{1.112}$$

Obviously, the variation of the penultimate term with respect to $t_2$ will identically give the energy.

Of course, the action is not unique. For an arbitrary smooth function $\phi : \mathbb{Q} \to \mathbb{R}$, the action

$$S[q^a] := S^1[q^a] - \int_{t_1}^{t_2}\frac{\mathrm{d}\phi}{\mathrm{d}t}\mathrm{d}t \tag{1.113}$$

is an equally good choice for the same problem: it affects the value of the integral but not the equations of motion. The energy in this formalism is therefore not unique as this change in the action shifts the latter it by a factor of $(\partial\phi/\partial t)$ [21].

### Formulating the Idea

We would now like to generalize the above procedure to a general relativistic spacetime, with the restriction that matter should be minimally coupled to gravity. While the following discussion will apply to arbitrary spacetime dimensions (actually, arbitrary $n+1$ spacetime dimensions where $n \geqslant 3$), it will be easier to imagine the foliations and boundaries in question if we specify the dimensions of the spacetime to be $3+1$.

We consider a compact region $D$ of a spacetime $(\mathcal{M}, g)$, foliated by spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ between times $t_1$ and $t_2$. The two-dimensional boundary $\partial\Sigma_t$ (of the leaf $\Sigma_t$) times the line time interval is denoted ${}^3B$. I.e. ${}^3B$ is a timelike boundary with topology $\partial\Sigma_t \times [t_1, t_2]$, and is defined by ([22])

$${}^3B := \bigcup_t \partial\Sigma_t. \tag{1.114}$$

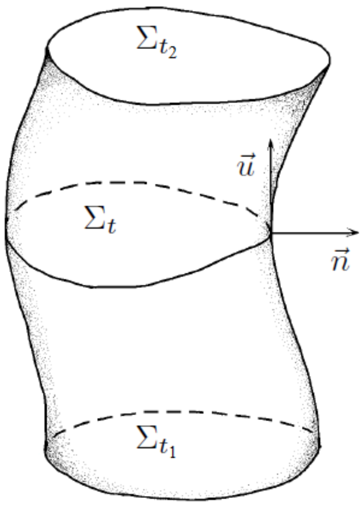


Figure 1.3: The foliation of the spacetime. *Credit: ref. [22]*

We denote the metric and extrinsic curvature on ${}^3B$ by $h_{ab}$ and $K_{ab}$, and those on $\Sigma_t$ by $\gamma_{ab}$ and $\Theta_{ab}$ respectively.

Consider an action functional $S^1$ in $h_{ab}$. Here the metric $h_{ab}$ sets the proper times between the initial and final configurations in $\Sigma_{t_1}$ and $\Sigma_{t_2}$, respectively. It therefore has a role analogous to the time interval in classical mechanics. The generalization of the energy term in this case will be an energy-momentum tensor. It will be defined analogously as the variation of a term that can be identified by varying $S^1$ with respect to a unit increase in proper time separation between $\partial\Sigma_t$ and its neighboring two-surface. This increase is measured orthogonally to $\partial\Sigma_t$. This means that the metrics of ${}^3B$, $\Sigma_{t_1}$ and $\Sigma_{t_2}$ are not kept fixed.

**Formalising the Idea**

The vector $n^a$ will be the spacelike unit normal vector to ${}^3B$ and $u^a$ the timelike unit normal vector to $\Sigma_t$ (see Figure 1.3). We would like the leaf $\Sigma_t$ to be "orthogonal" to ${}^3B$, meaning that

$$(u \cdot n)|_{{}^3B} = 0, \tag{1.115}$$

where, for the dot product to make sense, the vectors $n^a$ and $u^a$ are defined as four-dimensional vectors on the domain $D$.

The metric can be written under the usual metric decomposition:

$$ds^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i\mathrm{d}t)(\mathrm{d}x^j + \beta^j\mathrm{d}t). \tag{1.116}$$

The idea behind Brown and York's method is to calculate the variation of an appropriate action in GR and isolate the analogous term corresponding to the energy. The action proposed by Brown and York [20] is

$$\begin{aligned} S^1 =& \frac{1}{16\pi}\int_D \mathrm{d}^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\Sigma_{t_2}} \mathrm{d}^3x\sqrt{-\gamma}\Theta - \frac{1}{8\pi}\int_{\Sigma_{t_1}} \mathrm{d}^3x\sqrt{-\gamma}\Theta \\ &- \frac{1}{8\pi}\int_{^3B} \mathrm{d}^3x\sqrt{-h}K \,+\, S_M \end{aligned} \tag{1.117}$$

where $S_M$ is the matter action, which might also account for the cosmological constant action term. Now consider the evolution of the system from the hypersurface $\Sigma_{t_1}$ at $t_1$ to the hypersurface $\Sigma_{t_2}$ at $t_2$. The variation of the principal functional yields terms proportional to the variation of $g_{ab}$, which lead to the equations of motion as derived in Appendix D. Since now we allow the boundary metrics to vary, we have additional terms proportional to the $\delta h_{ab}$ and others proportional to $\delta\gamma_{ab}$. The variation of the principal action thus takes the form

$$\begin{aligned} \delta S^1 =& \text{(terms giving the equations of motions)} \\ &+ \text{(boundary terms coming from the matter action)} \\ &+ \int_{\Sigma_{t_1}}^{\Sigma_{t_2}} \mathrm{d}^3x\text{(terms proportional to } \delta\gamma_{ab}) \\ &+ \int_{^3B} \mathrm{d}^3x\text{(terms proportional to } \delta h_{ab}). \end{aligned}$$

To express the latter terms, we now consider variations due to an arbitrary change in boundary metrics $h_{ab}$ and $\gamma_{ab}$. To keep track of this, we will denote those variations by $\delta_h$ and $\delta_\gamma$[9] respectively. The variation of the term $\sqrt{-h}\,K$ gives

$$\delta_h(\sqrt{-h}K) = (\delta_h\sqrt{-h})K + \sqrt{-h}\,\delta_h K. \tag{1.118}$$

[9]Evidently, instead of using the notation $\delta_h x$ we could have used the less compact expression $(\delta x/\delta h_{ab})\,\delta h_{ab}$.

From equation (D.8), the first term is given by

$$(\delta_h \sqrt{-h})K = -\frac{1}{2}\sqrt{-h}h^{ab}K\delta h_{ab}. \tag{1.119}$$

The variation of the extrinsic curvature $K$ due to an arbitrary variation of the metric $h_{ab}$ is simply

$$\begin{aligned}\delta_h K &= \delta_h(h_{ab}K^{ab}) \\ &= K^{ab}\delta_h h_{ab} + h_{ab}\delta_h K^{ab}.\end{aligned} \tag{1.120}$$

We are then left with

$$\delta_h(\sqrt{-h}K) = -\frac{1}{2}\left(K^{ab} - Kh^{ab}\right)\delta_h h_{ab}. \tag{1.121}$$

Likewise, we have have

$$\delta_\gamma(\sqrt{-\gamma}\,\Theta) = -\frac{1}{2}\left(\Theta^{ab} - \Theta\gamma^{ab}\right)\delta_\gamma \gamma_{ab}. \tag{1.122}$$

The variation of the principal functional can thus be expressed as

$$\begin{aligned}\delta S^1 =&\text{(terms giving the equations of motions, including possible matter fields)} \\ &+ \frac{1}{16\pi}\int_{\Sigma_{t_2}} \sqrt{-\gamma}\left(\Theta^{ab} - \Theta\gamma^{ab}\right)\delta\gamma_{ab}\,\mathrm{d}^3x \\ &- \frac{1}{16\pi}\int_{\Sigma_{t_1}} \sqrt{-\gamma}\left(\Theta^{ab} - \Theta\gamma^{ab}\right)\delta\gamma_{ab}\,\mathrm{d}^3x \\ &+ \frac{1}{16\pi}\int_{{}^3B} \mathrm{d}^3x\sqrt{-h}\left(K^{ab} - Kh^{ab}\right)\delta h_{ab}.\end{aligned} \tag{1.123}$$

As we have mentioned, the action and its corresponding functional are not unique, and the general functional can be expressed as

$$S := S^1 - S^0, \tag{1.124}$$

where $S^0$ is some arbitrary function on the boundary $\partial D = \Sigma_{t2} \cup B \cup \Sigma_{t1}$. In the classical mechanics case we derived the energy from by differentiating the action with respect to the dynamical variable $t$. To generalize this, we look to define an *energy-momentum tensor* instead of an *energy*. To relate the energy-momentum tensor to the differentiation of our principal functional with respect to dynamical variable $h_{ab}$, recall that in General Relativity, given a matter action $S_M$, we define the associated energy-momentum tensor by

$$T_M^{ab} := -\frac{2}{\sqrt{|g|}} \frac{\delta S_M}{\delta g_{ab}}. \tag{1.125}$$

In analogy with the above expression, we define the energy-momentum tensor as

$$\begin{aligned} \tau^{ab} &:= -\frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}} \\ &= \frac{1}{8\pi} \left( K^{ab} - K h^{ab} \right) + \frac{2}{\sqrt{-h}} \frac{\delta S^0}{\delta h_{ab}}. \end{aligned} \tag{1.126}$$

While the tensor in (1.125) represents just the matter fields, $\tau_{ab}$ represents both the gravitational and matter fields. Note that because of the restriction $(u \cdot n)|_{^3B} = 0$, the metric on $^3B$ can be written as

$$ds^2_{^3B} = -\tilde{N}^2 \mathrm{d}t^2 + \sigma_{ij}(\mathrm{d}x^i + \tilde{\beta}^i \mathrm{d}t)(\mathrm{d}x^j + \tilde{\beta}^j \mathrm{d}t). \tag{1.127}$$

Having $\tau_{ab}$ in hand, we can use equation (C.81) to define a conserved current $J[\xi]$ associated with any Killing vector $\xi$ in the spacetime by

$$J_a[\xi] := \tau_{ab}\xi^b. \tag{1.128}$$

To define a conserved charge we should integrate the charge density component of the current over our spatial boundary. This means that the integral starts at the spatial boundary at $t_1$ and ends on the spatial boundary at $t_2$.

$$Q[\xi] = \int_{[t_1,t_2]\cap {^3B}} \mathrm{d}^2x \sqrt{\sigma} u^a \tau_{ab} \xi^b. \tag{1.129}$$

The above quantity is known as the “Brown-York quasi-local charge”. To understand what *quasi-local* means and why this quantity is quasi-local, first let us recall what we mean by *local* and *non-local* quantities. A *local* quantity at a particular point in spacetime can only depend on the values of other quantities at that particular spacetime point. By contrast, a *non-local* quantity at a particular point in spacetime can depend on the values of quantities at other points in spacetime. In fact, a non-local quantity can depend on the values of other quantities at every point in spacetime. A *quasi-local* quantity depends on the values of other quantities in a finite region in spacetime. For instance, the quantity $Q$ in the integral (1.129) depends on the values of the different quantities in the integrand at $t_1 \cap {}^3B$ and $t_2 \cap {}^3B$. It is therefore a quasi-local quantity.

Quasi-local charges are charges associated with closed $D-2$ surfaces. They are coordinate independent and do not depend on the choice of time slicing of the containing surface, but depend on the boundary conditions [21].

We are particularly interested in calculating the quasi-local energy and angular momentum in $n+1$ dimensions. Following the above discussion, the **Brown-York quasi-local energy** is given by

$$\boxed{E_{\mathrm{BY}} = \int \mathrm{d}^{n-1}x\sqrt{\sigma}u^a\tau_{ab}\chi^b,} \tag{1.130}$$

Likewise, the angular momentum in the $\phi$-direction is given by

$$\boxed{J_{\mathrm{BY}} = \int \mathrm{d}^{n-1}x\sqrt{\sigma}u^a\tau_{ab}\eta^b,} \tag{1.131}$$

where again $\eta = \partial_\phi$. The time slices are usually taken at the horizon and the boundary when $r$ goes to infinity since we would like to integrate over the spacetime region outside the black hole.

### Calculations for the Kerr Black Hole

As a quick application to the Brown-York formula, we will calculate the mass and angular momentum of the Kerr black hole solution given given by setting $q = 0$ in(1.52). The resulting metric is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{2mar\sin^2\theta}{\rho^2}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2, \end{aligned} \tag{1.132}$$

where

$$\Delta(r) = r^2 - 2mr + a^2, \tag{1.133}$$
$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \tag{1.134}$$

Note that the outer horizon of this metric is found as the largest root of $\Delta(r) = 0$. This gives

$$r_+ = m + \sqrt{m^2 - a^2}. \tag{1.135}$$

To divide our spacetime into spacelike hypersurfaces, we chose a time flow vector

$$\hat{t}^a = (1,0,0,0). \tag{1.136}$$

This leads to the lapse function

$$N = \sqrt{\frac{(r^2 + a^2\cos^2\theta)(a^2 - 2mr + r^2)}{\cos^2\theta a^4 - 2\cos^2\theta a^2 mr + \cos^2\theta a^2 r^2 + 2a^2 mr + a^2 r^2 + r^4}}.$$

Recall that the unit normal vector is given by

$$u = -N\mathrm{d}t, \tag{B.19}$$

The foliation metric given by

$$\sigma_{ab} = g_{ab} + u_a u_b \tag{B.29}$$

has a determinant given by

$$\sigma = \left(r^2 + a^2\cos^2\theta\right)\sin^2\theta\left(a^2 + r^2\right). \tag{1.137}$$

We can use a software like Maple to evaluate the expressions in (1.130) and (1.131). These expressions are the be evaluated at a surface of constant radial coordinate going to infinity. Direct evaluations on Maple give

$$M_{\mathrm{Kerr}} = m, \tag{1.138}$$

and

$$J_{\mathrm{Kerr}} = am. \tag{1.139}$$

You can easily note that the Kerr metric reduces to the Schwarzschild metric for $a = 0$. This is logical since the Schwarzschild solution, as we have mentioned, is the unique static spherically symmetric solution to the Einstein equation in asymptotically-flat spacetimes. Making the rotation parameter $a$ vanish will hence yield the Schwarzschild solution. Since the expression for the mass in (1.138) does not depend on $a$, the Brown-York energy gives the same result as the Komar integral in this case. As a matter of fact, we could show that the angular momentum obtained using the Komar integral will also yield the same result in (1.139). You might then wonder we spent some time to introduce two methods that are completely equivalent. In fact, they are not. As we will later see, in different spacetimes with different asymptotic topologies, the two methods can yield different expressions for the charges.

# Chapter 2

# Black Hole Thermodynamics

*"The law that entropy always increases — the second law of thermodynamics — holds, I think, a supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations, then so much the worse for Maxwell's equations. If it is found to be contradicted by observations — well, these experimentalists do bungle sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in humiliation." — Sir Arthur Eddington*

## 2.1 Cosmic Censorship Conjecture

One nice feature of classical (non-relativistic) physics is its power of predictability. Given the equations of motion, and any particular state, we can trace back the full history of the system. In general relativity, if a spacetime admits a Cauchy surface, we could predict the state of the universe at any time in the past or the future given relevant data on that surface [23].

We have seen in the case of the Schwarzschild solution that GR does present singularities (i.e. the singularity at the center of the Schwarzschild black hole). Furthermore, Penrose and Hawking have presented a set of theorems in the 1960's assuring that singularities are inevitable in GR [23]. These theorems predict singularities in two situations. The first one is in the future; the future singularities form by the gravitational collapse of stars and other massive bodies. The other situation is in the past, at the beginning of the current expansion of the universe [24]. This is conventionally thought of as what we call "the Big Bang". So the important thing is that singularities are not just a problem with the Schwarzschild solution, time-dependent solutions in GR will often end in singularities [11]. One cannot define the field equations where these singularities are and hence, as we have previously mentioned, these singularities do not belong to our manifold. General relativity then does not deal with these singularities and, given a past singularity, it cannot predict what will come out of it [24].

It might then seem that GR does not present the level of predictability that one would hope for. We can take solace however in a result formulated by Roger Penrose, which gives us a way around this.

**Strong Cosmic Censorship Conjecture.** In all physically reasonable spacetimes, aside from an initial singularity (such as the Big Bang), all singularities are hidden behind event horizons. Equivalently, this means that the spacetime is globally hyperbolic (cf. §A.3) [16].

Since we do not have access to information behind event horizons anyway, it seems that General Relativity presents a sufficient level of predictability for the "accessible" part of our universe. Of course, in a quantum theory of gravity there will be a level of unpredictability resulting from the quantum description, but that is another story, and one with which we shall not concern ourselves in this thesis.

The Cosmic Censorship Conjecture (CCC) is a *conjecture* because a precise proof of it remains elusive. It continues to be one of the most prominent issues in GR [11].

An important consequence of CCC was proposed by Hawking [23]. It states that, under the weak energy condition, the size of a black hole, as measured by the area of its event horizon, never shrinks, it only grows. First let us recall the **weak energy condition**. It signifies that for all timelike vectors $\xi^a$, the energy-momentum tensor satisfies [11]

$$T_{ab}\xi^a\xi^b \geq 0. \tag{2.1}$$

Physically this means that the energy density measured by an observer with velocity $\xi^a$ is positive.

The area theorem, as we will see, plays an important role in the laws of black hole physics. The reader can probably see an analogy between a property of the area dictated by the above theorem and entropy: both quantities cannot decrease. Of course, this is nothing but a coincidence so far. As we will discuss below, our discussion of black holes until now does not qualify them to be considered "thermodynamical systems".

## 2.2 The Penrose Process and Black Hole Mechanics

From our discussion on black holes so far, it appears that no objects can be extracted from the black hole past the event horizon. We will see however that energy *can* be extracted from a black hole via what is called "the Penrose process".

Imagine a Kerr black hole and a person holding a ball hovering over it. The spacetime will have the usual Killing vectors $\chi = \partial_t$ and $\eta = \partial_\phi$, and the horizon Killing vector $\xi = \chi + \Omega\eta$. We associate to the the system of (the person+the ball) energy and angular momentum given by the quantities:

$$E = -p^a \chi_a, \tag{2.2}$$

and

$$L = p^a \eta_a. \tag{2.3}$$

The negative sign in expression (2.2) is there because, at infinity, both $p^a$ and $\chi^a$ are timelike, and we would like to have a positive total energy there. Inside the ergosphere however, $\chi^a$ is spacelike, and we could imagine a situation where

$$E = -p^a \chi_a < 0. \tag{2.4}$$

Of course, the person's total energy would become positive once she is out of the ergosphere. Let us denote the energies of the person and the ball by $E_1$ and $E_2$ respectively (such that $E = E_1 + E_2$). The person can arrange to throw the ball into the black hole with energy

$$E_2 < 0. \tag{2.5}$$

Penrose has shown that it is possible to do so and then follow a geodesic trajectory outside the stationary limit surface, where the person's energy will necessarily be positive. In this particular situation, the ball must be thrown in such a way that it has a negative angular momentum $L_2$. This just means that its angular momentum is in the opposite direction to that of the black hole's. Ultimately, after the person crosses the stationary limit surface, she will have energy

$$E_1|_{\text{outside}} > E_0. \tag{2.6}$$

This means that she left the ergosphere with more energy than she had inside. Since the Kerr spacetime is stationary, and energy is conserved, the person must have received energy from somewhere. In our example we only have three entities: the black hole, the person and the ball. That extra energy must then have come from the black hole.

After the ball is thrown into the black hole, the total mass and angular momentum of the black hole must then change by amounts of

$$\delta M = E_2, \tag{2.7}$$

and

$$\delta J = L_2. \tag{2.8}$$

$\delta M$ is the energy extracted from the black hole, and $\delta J$ is the change imposed on the black hole's angular momentum by throwing inside it a ball with a negative angular momentum.

It is insightful to show that this process does not violate the area theorem. To do so, we start by writing the induced metric on the outer horizon $\mathcal{H}$ ($\mathrm{d}r = 0$, $r = r_+$):

$$h_{ij} = (r_+^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \left[\frac{(r_+^2 + a^2)^2\sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right]\mathrm{d}\phi^2. \tag{2.9}$$

The area of the horizon is given by

$$\begin{aligned}\mathcal{A} &= \int_{\mathcal{H}} \sqrt{|h|}\,\mathrm{d}\theta\mathrm{d}\phi \\ &= \int_{\mathcal{H}} (r_+^2 + a^2)\mathrm{d}\theta\mathrm{d}\phi \\ &= 4\pi(r_+^2 + a^2).\end{aligned} \tag{2.10}$$

Plugging equation (1.135) into this, we have

$$\begin{aligned}\mathcal{A} &= 4\pi\left[\left(m + \sqrt{m^2 - a^2}\right)^2 + a^2\right] \\ &= 4\pi\left(m^2 + m^2 - a^2 + 2m\sqrt{m^2 - a^2} + a^2\right) \\ &= 8\pi\left(m^2 + \sqrt{m^4 - m^2a^2}\right).\end{aligned} \tag{2.11}$$

Identifying this with equation (1.139), we get

$$\mathcal{A} = 8\pi\left(m^2 + \sqrt{m^4 - J^2}\right). \tag{2.12}$$

We can hence write the change in $\mathcal{A}$ as a function of the aforementioned $\delta M$ and $\delta J$:

$$\delta\mathcal{A} = 8\pi \left[ \left( \frac{2m^3}{\sqrt{m^4 - J^2}} + 2m \right) \delta M - \frac{J}{\sqrt{m^4 - J^2}} \delta J \right]. \tag{2.13}$$

Using the explicit form of $\Omega_H$ in equation (1.62), along with some straightforward but messy calculations, we arrive at

$$\delta\mathcal{A} = 8\pi \frac{a}{\Omega_H \sqrt{m^2 - a^2}} (\delta M - \Omega_H \delta J). \tag{2.14}$$

It is not yet obvious that this expression is strictly positive (i.e. that the area can not decrease). To see this, recall that outside the black hole $p^a$ and the horizon Killing vector $\xi^a$ are both timelike, thus

$$\xi_a p_2^a < 0. \tag{2.15}$$

By expanding $\xi^a$ and sticking to the definition in (2.3) for $L_2$, we find that

$$E_2 > \Omega_H L_2, \tag{2.16}$$

which can be recast as

$$\delta M > \Omega_H \delta J. \tag{2.17}$$

This directly shows that the expression (2.14) is strictly positive. While we have not given a proof for the area theorem before, we can now see what happens when energy is either added to or extracted from the black hole. Equation (2.12) implies that any increase in the black hole's energy will result in an increase in the area, while equation (2.14) shows that any energy extracted from the black hole will also result in a positive change in the area.

In deriving equation (2.14), we have arrived at the key equation of this thesis! To see what exactly is so striking about this equation, let us rearrange it into

$$\delta M = \frac{\kappa}{8\pi} \delta\mathcal{A} + \Omega_H \delta J, \tag{2.18}$$

where we used the fact that the (outer) surface gravity of the Kerr black hole is given by

$$\kappa = \frac{\sqrt{m^2 - a^2}}{2m\left(m + \sqrt{m^2 - a^2}\right)}. \tag{2.19}$$

Equation (2.18) made people first start thinking about a possible correspondence between black holes and thermodynamics due to its resemblance to the first law of thermodynamics [11]

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V, \tag{2.20}$$

where $P\mathrm{d}V$ represents the work done by the system. It is reasonable to think of the term $\Omega_H \delta J$ as the work we do on the black hole by throwing the ball into it. Since $M = m$ is already the black hole energy, there is an analogy between the following terms [11]:

$$\begin{aligned} S &\to \mathcal{A}/4 \\ T &\to \kappa/2\pi. \end{aligned} \tag{2.21}$$

Aside from the analogy between black holes and the first law of thermodynamics, we can note that the area theorem looks quite similar to the fact that the entropy of a closed thermodynamical system cannot decrease – i.e. the second law of thermodynamics. This also goes well with the analogy given above between the entropy and the black hole area (divided by 4).

If we wanted to be a little more generous (or pushy) we could also claim a resemblance with the zeroth law. The zeroth law states that the temperature is constant throughout a system in thermal equilibrium. A black hole in thermal equilibrium will have settled into a stationary state, having constant surface gravity across the horizon. This goes well with the second analogy in (2.21).

Analogy between laws of black holes and the third law of thermodynamics is a bit more complex, mainly because there are several versions of the third law of thermodynamics. The weaker (Nernst) form of the third law is that the temperature of a system cannot be reduced to 0 by a finite number of operations [25]. Werner Israel showed [26] that the surface gravity of a black hole cannot be reduced to 0 by a finite sequence of operations. In this sense, there *is* an analogy between black hole laws and the third law of thermodynamics. However, there is no such analogy with the stronger (Planck) form of the third law of thermodynamics that the entropy of a system becomes 0 when the temperature goes to 0 [25]. But in the end, the third law is not really a law in the sense that other certain systems actually violate it as well [11].

While we have discussed the first law of black hole mechanics for a Kerr black hole, it can obviously be reduced to the case of a Schwarzschild black hole by setting $a = 0, J = 0$. This is of course very easy to check. It is also worth noting that the first law does apply to a charged, rotating Kerr-Newman black hole. Accounting for charges and an electric potential turns out to be analogous to accounting for the number of particles $N$ and chemical potential $\mu$ of a thermodynamic system, where the first law becomes

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N \tag{2.22}$$

It is straightforward (but very tedious) to check that the more general form of (2.18) which accounts for electric charge and potential (i.e. the first law for the Kerr-Newman solution for example) is simply

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \Omega_H \delta J + \Phi \mathrm{d}Q. \tag{2.23}$$

Here the electric potential is $\Phi = -Q/r$ as usual, but evaluated at the horizon like the rest of the other intensive quantities. Equation (2.23) is the generic form of this formula in four dimensions. Note that in extra dimensions there are additional axes which the black hole may have an angular momentum about. The first law then incorporates the sum of quantities $\Omega_i \mathrm{d}J_i$ around all these axes,

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \sum_i \Omega_i \delta J_i + \Phi \mathrm{d}Q.} \tag{2.24}$$

While everything looks so easy and straightforward in 4-dimensional, asymptotically flat spacetimes, we will see that the arrival at an equivalent formula in spacetimes with different topologies and dimensions can be more challenging (cf. §3.3).

Having pointed out the striking similarities between certain laws in black hole physics and the laws of thermodynamics, it is worth noting that so far, these are merely *similarities*. To see this more clearly, recall that temperature is defined as a quantity which two systems have different values of if they are exchanging heat. So far we have claimed that black holes can only absorb but not emit energy, hence it cannot "exchange" energy (or heat) with another system. It is therefore paradoxical to claim that the quantity $\kappa/2\pi$ is equivalent to a black hole temperature. In retrospect, the laws on the black holes side have nothing thermal about them (yet), and we could refer to them as the laws of black hole *mechanics*.

## 2.3 Black Hole Thermodynamics

A huge step forward was taken when Stephen Hawking discovered that black holes are not really black, but rather emit radiations [27]. This was done by applying quantum field theory in curved spacetimes to the neighborhood of the event horizon. It was the first indication that a black hole does have a temperature, one which is associated with these *Hawking radiations*, and which it turned out to be given by

$$\boxed{T = \frac{\kappa}{2\pi}}. \tag{2.25}$$

A different derivation of the black hole radiation effect was also presented by Maulik Parikh and Frank Wilczek in [28], wherein particles escape from the event horizon via quantum tunneling. This makes sense; our understanding of the outside region being forbidden for a particle inside the black hole is completely classical. It is natural to assume that it may be overcome via quantum tunneling.

Not only does equation (2.25) allow us to think of black holes as thermodynamical systems, but when plugged into (2.18) along with the analogy $S \to \mathcal{A}/4$, we magically retrieve the exact usual form of the first law of thermodynamics

$$\mathrm{d}E = T\mathrm{d}S + \text{work done on the (black hole) system}, \tag{2.26}$$

where the black hole mass (or energy) is now denoted $E$ for dramatic effect.

We have now argued that the resemblance between (2.18) and (2.20) is not merely in terms of how "similar" they look, but that the terms in both equations have the same physical meaning, except for the entropy which is the last piece of the puzzle. Since we have already agreed that black holes have temperatures, it makes sense to think about what entropy a black hole system may possess. In the following few paragraphs, we will give a qualitative explanation of how this was done.

We start by recalling that on the macroscopic level, a black hole can be solely described by its mass, angular momentum and electric charge, without the need for knowledge about any internal parameters. A black hole with given values for $M$, $J$ and $Q$ has a number of different possible internal configurations. If we neglect quantum effects, then the number of these configurations would become infinite since a black hole may be formed by an infinite number of infinitely small particles [29]. However, Bekenstein noted that one would have to restrict the Compton wavelength of these particles to be less than the radius of the black hole, rendering the number of possible internal configurations finite [29].

Let $\sigma \mathrm{d}M\, \mathrm{d}Q\, \mathrm{d}^3 J$ be the number of internal configurations of a black hole in the range of $M$

to $M+\mathrm{d}M$, $Q$ to $Q+\mathrm{d}Q$ and angular momentum in the element $\mathrm{d}^3J$ about a given angular momentum $\vec{J}$. Since by the "no-hair" theorem one has no knowledge of the internal state of the system, all configurations are equiprobable, and the entropy is given by ([29])

$$S = -\Sigma_i p_i \ln p_i = \ln \sigma. \tag{2.27}$$

The entropy can also be described in terms of the initial states that give rise to a black hole in the aforementioned ranges. Let $\{|\alpha_i\rangle\}_i$ be a complete orthonormal basis of initial states. Let $f_i V \mathrm{d}^3V\, \mathrm{d}M\, \mathrm{d}Q\, \mathrm{d}^3J$ be the probability that the state $|\alpha_i\rangle$ gives rise to the black hole. Here $V$ is a normalization volume and the black hole is in the element $\mathrm{d}^3P$ about $\vec{0}$. Then the probability for a certain sate $|\alpha_i\rangle$ to give rise to the black hole is

$$q_i = \frac{f_i}{\Sigma_i f_i}. \tag{2.28}$$

The total entropy is also given by

$$S = \Sigma_i q_i \ln q_i. \tag{2.29}$$

The entropy has to be a function of only $M$, $J$, $Q$, and the following properties [29]:

1. It always increases when matter or radiation goes into the black hole.
2. When two (or subsequently more) black holes collide together, the resulting entropy of the new black hole is bigger than the sum of the entropies of the original holes.

It turns out that this needs to be a monotonic function $S(\mathcal{A})$, with

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\mathcal{A}^2} \geq 0. \tag{2.30}$$

The simplest such function is

$$c\,\mathcal{A}, \tag{2.31}$$

with $c$ some constant. The black hole temperature can be defined in an analogous manner to the classical thermodynamics temperature via

$$T^{-1} = \left(\frac{\partial S}{\partial E}\right)_{J,Q}. \tag{2.32}$$

With this formula, Hawking was able to verify that the correct value for $c$ is in fact $1/4$.

So in conclusion, black holes are not really black, they emit radiations. This radiation allows us to define the temperature and calculate the entropy of black holes. This leads to black holes having exact versions of the first and second laws of thermodynamics. Having identified the temperature with a constant times the surface gravity and the entropy with a constant times the surface area of the horizon, we can now boldly write the first law of black hole thermodynamics (for the Kerr-Newman case) as

$$\boxed{\mathrm{d}M = T\mathrm{d}S + \Omega\mathrm{d}J + \Phi\mathrm{d}Q}. \tag{2.33}$$

From this, we get the following equations of state which are analogous to those for intensive quantities in conventional thermodynamics:

$$\begin{aligned} T &= \frac{\partial M}{\partial S}, \\ \Omega &= \frac{\partial M}{\partial J}, \\ \Phi &= \frac{\partial M}{\partial Q}. \end{aligned} \tag{2.34}$$

It was shown my Smarr [30] that for Kerr-Newman black holes there is another equation worth noting:

$$M^2 = \frac{1}{4\pi}S + 16\pi S\left(J^2 + \frac{1}{4}Q^2\right) + \frac{1}{2}Q^2. \tag{2.35}$$

This, along with Euler's theorem for homogeneous functions, leads to another set of fundamental equations. To see this, first let us recall what we mean by "homogeneous function".

**Definition 2.1.** Let $S \subset \mathbb{R}^n$. Then a function of $n$ variables $f : S \to \mathbb{R}$ is said to be **homogeneous of degree $k$** if $\forall\,(x_1, x_2...x_n) \in S$, $\forall\,\lambda \in \mathbb{R}^*$, we have $(\lambda x_1, \lambda x_2, ...\lambda x_n) \in S$ and

$$f(\lambda x_1, \lambda x_2, ...\lambda x_n) = \lambda^k f(x_1, x_2...x_n). \tag{2.36}$$

We can now state Euler's theorem as follows:

**Theorem 2.1. (Euler's Theorem for Homogeneous Functions)** Given the above assumptions, then $f$ is homogeneous of degree $k$ if and only if

$$\sum_{i=1}^{n} x_i \partial_i f\left(x_1, \ldots, x_n\right) = k f\left(x_1, \ldots, x_n\right), \quad \forall\left(x_1, \ldots, x_n\right) \in S. \tag{2.37}$$

Now, from equation (2.35), we see that $M$ is a function of $S, J$ and $Q$. We will actually use a trick here and assume that $M$ is actually function of $\frac{1}{2}Q$. This works fine with (2.35) since $Q^4$ is multiplied by $\frac{1}{4}$. We did not really show if $M(S, J, Q/2)$ is homogeneous. However, we can make use of Euler's theorem and suppose that it is homogeneous of degree $k$, then check if our assumption works out. Thus, using the set of equations (2.47) we write

$$
\begin{aligned}
kM &\stackrel{?}{=} S\frac{\partial M}{\partial S} + J\frac{\partial M}{\partial J} + \frac{1}{2}\frac{\partial M}{\partial Q} && (2.38)\\
&\stackrel{?}{=} TS + \Omega J + \frac{1}{2}\Phi Q. && (2.39)
\end{aligned}
$$

It is straightforward to show that this is in fact true for $k = \frac{1}{2}$, leading to

$$
\frac{1}{2}M = TS + \Omega J + \frac{1}{2}\Phi Q. \tag{2.40}
$$

Differentiating the above equation and using the first law (2.33),

$$
\begin{aligned}
\frac{1}{2}\mathrm{d}M &= T\mathrm{d}S + S\mathrm{d}T + \Omega\mathrm{d}J + J\mathrm{d}\Omega + \frac{1}{2}\Phi\mathrm{d}Q + \frac{1}{2}Q\mathrm{d}\Phi\\
&= \frac{1}{2}\mathrm{d}M - \frac{1}{2}\Phi\mathrm{d}Q + S\mathrm{d}T + J\mathrm{d}\Omega + \frac{1}{2}Q\mathrm{d}\Phi && (2.41)
\end{aligned}
$$

From this, we get the following relations for the extensive quantities of the first law:

$$
\begin{aligned}
\left(\frac{\partial M}{\partial T}\right)_{\Omega,\Phi} &= -2S && (2.42)\\
\left(\frac{\partial M}{\partial \Omega}\right)_{T,\Phi} &= -2J && (2.43)\\
\left(\frac{\partial M}{\partial \Phi}\right)_{\Omega,T} &= -Q. && (2.44)
\end{aligned}
$$

In continuing the analogy with classical thermodynamics, we may define the Gibbs energy of the black hole. Recall that in classical thermodynamics the Gibbs energy is defined by

$$
G = E - TS - (-PV). \tag{2.45}
$$

For the charged rotating black hole this becomes

$$G = M - TS - \Omega J - \Phi Q. \tag{2.46}$$

Then we have another set of important thermodynamical relations for the intensive quantities in the first law,

$$\begin{aligned} S &= -\left(\frac{\partial G}{\partial T}\right)_{\Omega,\Phi}, \\ J &= -\left(\frac{\partial G}{\partial \Omega}\right)_{T,\Phi}, \\ Q &= -\left(\frac{\partial G}{\partial \Phi}\right)_{T,\Omega}. \end{aligned} \tag{2.47}$$

Having laid out the first law of thermodynamics and the above relations for intensive and extensive quantities that appear in it, it is astonishing that complex entities like black holes that emerge from General Relativity can be described via these simple thermodynamical relations, which only rely on their external parameters. Black hole thermodynamics are an important tool for understanding and developing a theory for quantum gravity [31]. Its study has lead to the development of one of the most active areas of theoretical physics research today: the AdS/CFT correspondence [2] which we will discuss in the following chapter.

# Chapter 3

# Anti-de Sitter Spacetime

## 3.1 The Anti-de Sitter Spacetime

All the metrics that we have presented in the previous chapters have been asymptotically flat. This is due to these metrics being solutions of the Einstein equation with no cosmological constant,

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}. \tag{3.1}$$

Of course, there is nothing to stop us from deriving solutions for the complete Einstein equation with cosmological constant

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}. \tag{3.2}$$

In vacuum, this gives

$$R_{ab} - \frac{1}{2} g_{ab} R = -\Lambda g_{ab}, \tag{3.3}$$

which you can easily verify leads to

$$R = \frac{D\Lambda}{\frac{D}{2} - 1}, \tag{3.4}$$

where $D$ is the dimension of the spacetime manifold. We will mostly be concerned with the cases where $D \geq 4$, for which the curvature scalar's signature is determined by $\Lambda$.

A spacetime for which $\Lambda$ is positive is called a **de Sitter spacetime**. It is named after Dutch mathematician Willem de Sitter who, in 1917, found a solution to the Einstein equation with a positive cosmological constant [32]. The de Sitter spacetime is a significant solution because it describes an expanding universe - which happens to be the case of the one we live in. The existence of a positive cosmological constant is the simplest representation of dark energy in the so-called Standard Model of Cosmology [33].

In contrast, a spacetime for which $\Lambda$ is negative is called an **anti-de Sitter spacetime** (AdS). This class of spacetimes is the one with which we will concern ourselves for the rest of the thesis. The question of why we are concerned with spacetimes that do not describe our universe is of course an important one – one which we will address soon enough.

The constant curvature property in equation (3.3) leads to the following form of the Riemann tensor ([34]):

$$R_{abcd} = -\frac{1}{\ell^2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right), \tag{3.5}$$

where $\ell$ is called **the AdS radius**. Contracting equation (3.5) leads to

$$R_{ab} = -\frac{D-1}{\ell^2}g_{ab}, \tag{3.6}$$

and

$$R = -\frac{D(D-1)}{\ell^2}. \tag{3.7}$$

Combining this with (3.3) leads to the important formula

$$\Lambda = -\frac{(D-1)(D-2)}{2\ell^2}. \tag{3.8}$$

The solution to equation (3.3) with negative cosmological constant turns out to be the metric describing a hyperboloid sheet. A $(D-1)$-dimensional hyperboloid sheet is given by the quadratic equation

$$-(X^0)^2 - (X^D)^2 + \sum_{i=1}^{D-1} X_i^2 = -\ell^2. \tag{3.9}$$

This is a hyperboloid surface with outer radius $\ell$, explaining the name "AdS radius". The AdS spacetime here has topology $\mathbb{R} \times S^{D-1}$.

To find a solution for the Einstein equation in 4 dimensions with a negative cosmological constant, we assume that the metric takes a similar form to those in Chapter 1 with an undetermined ansatz function $f(r)$ ([35]):

$$ds^2 = -f(r)^2 \mathrm{d}t^2 + f(r)^{-2} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{3.10}$$

The resulting Ricci tensor is diagonal, with components

$$R_{tt} = -f^4 R_{rr} = f^3 \left( f'' + \frac{2f'}{r} + \frac{f'^2}{f} \right), \tag{3.11}$$

and

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} = (1 - f^2 - 2ff'r)\sin^2\theta. \tag{3.12}$$

The Einstein equation in vacuum in four dimensions obviously reduces to

$$R_{ab} = \Lambda g_{ab}. \tag{3.13}$$

The $tt$ and $rr$ components lead to the following constraint for $f(r)$:

$$f'' + \frac{2f'}{r} + \frac{f'^2}{f} = \frac{-\Lambda}{f}. \tag{3.14}$$

The $\theta\theta$ and $\phi\phi$ components lead to a second constraint,

$$1 - 2ff'r - f^2 = \Lambda r^2. \tag{3.15}$$

We can use Mathematica to determine the ansatz given the above constraints, which yields

$$f(r) = 1 + \frac{r^2}{\ell^2}, \tag{3.16}$$

with

$$\ell^2 = \frac{-3}{\Lambda}. \tag{3.17}$$

The generic anti-de Sitter vacuum metric in $D$ dimensions is given by ([35])

$$-\left(1+\frac{r^2}{\ell^2}\right)\mathrm{d}t^2+\frac{1}{1+\frac{r^2}{\ell^2}}\mathrm{d}r^2+r^2\mathrm{d}\Omega_{D-2}^2, \tag{3.18}$$

where $\mathrm{d}\Omega_{D-2}^2$ is the metric on the unit $(D-2)$-sphere.

An important notion of asymptotically AdS spacetimes is that of *conformal flatness*. Let us start by defining what this is.

**Definition 3.1.** A Riemannian or pseudo-Riemannian manifold $\mathcal{M}$ is **conformally flat** if and only if $\forall\, x \in \mathcal{M}\, \exists$ a neighbourhood $\mathcal{V}$ of $x$ and a smooth function $\Omega(x) : \mathcal{M} \to \mathbb{R}^*$ such that the conformal metric $\tilde{g}_{ab} = \Omega^2(x) g_{ab}$ has a vanishing Riemann curvature tensor on $\mathcal{V}$.

Given the previous definition, we have the important following theorem.

**Theorem 3.1. (The Weyl–Schouten Theorem)** A Riemannian or pseudo-Riemannian manifold of dimension $D \geq 4$ is conformally flat if and only if its Weyl tensor vanishes.

If we multiply the metric in (3.18) by a factor of $(1-r^2/\ell^2)^2/(1+r^2/\ell^2)^2$, the resulting metric is

$$-\mathrm{d}t^2+\left(\frac{2}{1+r^2/\ell^2}\right)^2(\mathrm{d}r^2+r^2\mathrm{d}\Omega_{D-2}^2). \tag{3.19}$$

Specifically at $r = \ell$ we recover the flat Minkowski metric in $D$ dimensions. We can thus see that the metric is conformally flat.

The final notion that we will discuss in this section is that of an asymptotically anti-de Sitter spacetime.

**Definition 3.2.** A $D$-dimensional spacetime $(\mathcal{M}, g_{ab})$ is said to be **weakly asymptotically anti-de Sitter** if there exists a spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ with boundary $\partial\tilde{\mathcal{M}}$ and a diffeomorphism from $\mathcal{M}$ onto $\mathcal{M} - \partial\tilde{\mathcal{M}}$ such that
1. there exists a function $\Omega : \tilde{\mathcal{M}} \to \mathbb{R}$ such that $\tilde{g}_{ab} = \Omega^2 g_{ab}$ on $\mathcal{M}$,
2. $\partial\tilde{\mathcal{M}}$ is topologically $\mathbb{R} \times S^{D-2}$, and on $\Omega(x) = 0$, $\forall\, x \in \partial\tilde{\mathcal{M}}$, and
3. $g_{ab}$ satisfies the Einstein equation with negative cosmological constant and energy-momentum tensor $T_{ab}$, where $\Omega^{-3} {T_a}^b$ admits a smooth limit to $\partial\tilde{\mathcal{M}}$ [19].

**Definition 3.3.** A spacetime is said to be **asymptotically anti-de Sitter** if, in addition to Definition 3.2, the boundary $\partial\tilde{\mathcal{M}}$ is conformally flat.

## 3.2 The AdS/CFT Correspondence

### 3.2.1 Background

We will divert our focus from classical General Relativity for a short while to discuss Quantum Field Theory and String Theory before we present the AdS/CFT correspondence in the following section. Our aims is to give a self-contained discussion, with only preliminary knowledge of Quantum Mechanics (and not Quantum Field Theory) required from the reader. Moreover, this section is not meant to give a formal introduction to the topics that will be discussed. Our goal here is to give a non-mathematical (as much as possible) description to the reader of certain concepts that will be used in following parts of the thesis.

A **field** is a tensor (including scalars and vectors) having a value at each point in spacetime. A **field theory** is a theory whose dynamical variables are fields and not point particles. General Relativity for example is of course a field theory, since the dynamics are represented by the metric tensor field. It is however a *classical* field theory. A **Quantum Field Theory** is a field theory whose fields are quantized.

Conventional Quantum Mechanics, as the reader may know, is non-relativistic. To study interactions at very high energies and very small scales we then require a relativistic description of quantum physics. The natural thing to do is to attempt to formulate a relativistic theory of Quantum Mechanics itself. Relativistic Quantum Mechanics does in fact exist but it has several inconsistencies rendering it impractical for use in the study of Particle Physics. For instance, consider the amplitude of a particle traveling from a point $\vec{x}_0$ to another point $\vec{x}_1$. Using the relativistic formula for energy

$$E = \sqrt{p^2 + m^2}, \tag{3.20}$$

this amplitude is given by

$$\begin{aligned} U(t) &= \langle \vec{x}_1 | e^{-it\sqrt{p^i p_i + m^2}} | \vec{x}_0 \rangle \\ &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 p e^{-it\sqrt{p^i p_i + m^2}} \cdot e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_0)} \\ &= \frac{1}{2\pi^2 |\vec{x}_1 - \vec{x}_0|} = \int_0^{+\infty} \mathrm{d}p\, p \sin\left(p|\vec{x_1} - \vec{x_0}|\right) e^{-it\sqrt{p^i p_i + m^2}}. \end{aligned} \tag{3.21}$$

If we look at the asymptotic behavior of this integral well outside the light-cone for $x^i x_i \gg t^2$, we can show that ([36])

$$U(t) \sim e^{-m\sqrt{x^2-t^2}}. \tag{3.22}$$

The probability that a particle exists outside the light-cone is hence non-zero, which breaks causality. The use of Quantum Field Theory solves this and other inconsistencies in relativistic Quantum Mechanics [36].

The calculation of interaction and scattering amplitudes in Quantum Field Theory is non-exactly solvable. It would then be useful to express these amplitudes in terms of a perturbation series. Particularly, these perturbation series are expressed as expansions in the *coupling coefficient* of the theory. Coupling coefficients are constants of the theory that describe the strength of the force. In this manner, the $n^{\text{th}}$ term in the series is proportional to the $n^{\text{th}}$ power of the coupling coefficient.

For quantum electrodynamics this coefficient is the fine-structure constant $\alpha \approx 1/137$. Since $\alpha^n$ decreases with each term in the series, the perturbation series may converge. More importantly, we can take a cut-off of the first few terms and neglect the following terms since, if finite, they will be of less significant value.

For quantum chromodynamics however the story is different. The coupling coefficient is not constant. It becomes very large at very low energies, and perturbation theory can no longer be used in QCD calculations.

So far, Quantum Field Theory (QFT) gives very good description of three of the four fundamental forces of nature: the strong force, the electromagnetic force and the weak force. However, it has failed in providing a full quantum description of gravity. By this we mean that there are ways of quantizing General Relativity and using the resulting QFT as an effective low-energy theory. However, at high energies this gives rise to divergences which cannot be eliminated by known regularization techniques.

In comes String Theory. String Theory takes another approach at quantizing physics. Instead of point-like fundamental particles, it considers string-like particles and quantizes those. So in essence, String Theory is a theory of how strings propagate and interact with each other. Those strings can vibrate in different ways and one of those vibrational states gives rise to the graviton: the elementary gauge boson that would be the force carrier for gravity. Thus String Theory provides a quantum description of gravity. In fact, currently it is *the* prime candidate for a quantum theory of gravity [1]. Like Quantum Field Theory, String Theory may incorporate perturbation theory as well. When the coupling coefficient is weak enough, a solution may be approximated by the first order of the perturbation theory, which is simply a classical description of gravity. In this sense, studies of certain black hole solutions in General Relativity is related to studies of solutions in String Theory in a certain limit.

### 3.2.2 Presentation of the Conjecture

We have seen in the previous section that, although we live in a de Sitter universe, we can construct well defined anti-de Sitter spacetimes which represent solutions to the Einstein field equation in various dimensions. It is now time to talk about why anti-de Sitter spacetimes are important.

The importance of AdS spacetimes lies in what is called the *AdS/CFT correspondence*. "CFT" stands for *conformal field theory*. This is a class of quantum field theories which are invariant under conformal transformation of the metric [37]. The AdS/CFT correspondence is a duality between String Theory in anti-de Sitter spacetime and Quantum Field Theory in a conformally flat spacetime [1]. The quantum field theory may be thought of as being defined on the boundary of the anti-de Sitter spacetime, which we already saw is conformally flat. In a particular limit, if the field theory is strongly coupled, the gravity formulation on the String Theory side is weakly curved and can be approximated by a classical solution [1]. This leads to a map between a type of strongly-coupled conformal field theory in four dimensions having gauge symmetry group SU($N$) and classical black hole solutions in five-dimensional anti-de Sitter spacetime. In this case the aforementioned limit is when $N \to \infty$. We will see in following sections how this translates into the calculations in more details.

For now, the important result is that the AdS/CFT correspondence, roughly speaking, claims that

Strongly-coupled gauge theory at finite temperature =
gravitational theory in AdS black hole [32].

For instance, the Brown-York energy momentum given by (1.126), when calculated in an anti-de Sitter bulk spacetime, can be interpreted as the expectation value of the energy-momentum tensor resulting from a quantum effective action in the conformal field theory defined on the anti-de Sitter boundary [38]:

$$\langle \hat{T}^{ab} \rangle = -\frac{2}{\sqrt{-h}} \frac{\delta S_{\text{eff}}}{\delta \gamma_{ab}}, \tag{3.23}$$

where $\gamma_{ab}$ is the metric of the conformal field theory.

The original paper [2] that proposed the AdS/CFT correspondence is a 1997 paper by Juan Martín Maldacena. As of the writing of this section, it is the most highly cited paper in the field of theoretical high energy physics [39].

The duality is a powerful tool in studying quantum gravity. Since we understand conformal field theories much better than we understand quantum gravity, we can use our knowledge of a conformal field theory and the AdS/CFT duality to gain insight into quantum gravity

[40]. Likewise, we can use the equivalence to perform calculations in the anti-de Sitter spacetime that would be more complicated to perform in the realm of Conformal Field Theory [40].

One of the important applications of the AdS/CFT correspondence is its use to perform certain calculations related to color confinement in quantum chromodynamics [40]. Color confinement is the phenomenon that particles carrying color charge cannot be isolated and thus cannot be observed individually in nature. The AdS/CT correspondence leads to a duality between phase transitions in black hole solutions and confinement-deconfinement phase transitions [41]. The theory is also important in studying other strongly-coupled systems in condensed matter physics [40]. There is a particular interest in understanding strongly-correlated systems at finite temperature. This turns out to be very difficult with presently-known condensed matter computation techniques [40]. This gives the AdS/CFT correspondence a place in the world of applied physics as well. The bulk studies will naturally concern a black hole in AdS spacetime. We usually would like to keep the system at a finite chemical potential, which turns out to correspond to using charged black hole models with electrical potential.

In our work, we will be interested in using the AdS/CFT correspondence to study two particular phenomena: the conformal anomaly and the Casimir energy. The first will be discussed in §3.2.3 and the latter in §3.4. In Chapter 4 we will then use data from the bulk gravity theory to calculate the conformal anomaly and Casimir energy of the corresponding CFT and compare them to calculations of anomaly and Casimir energy calculated in the CFT itself.

### 3.2.3 Conformal Anomaly

In section §3.2 we mentioned that conformal field theories are a class of quantum field theories that remain invariant under conformal transformation of the metric. When we say "the theory remains invariant" we particularly mean that its action, which generates the equations of motion, stays invariant under said transformations.

Like any QFT, CFT's sometimes present divergent observable quantities which need to be *regularized* in order to obtain a finite value. A classical example of a regularization is calculating the mass of a point-charge like the electron by taking into account its energy in the electrostatic field (i.e. the electromagnetic mass). The calculation takes the form

$$m_{\mathrm{EM}} = \int \frac{1}{2} E^2 \mathrm{d}^3 x = \frac{Q^2}{8\pi r_e}, \tag{3.24}$$

with $r_e$ being the electron's radius. By taking the classical limit $r_e \to 0$, $m_{\mathrm{EM}}$ obviously diverges, which contradicts the electron having an experimental finite mass. Regularizing the value of $m_{\mathrm{EM}}$ consists of admitting that the classical theory breaks down at the fundamental microscopic level where the value of the electron's radius becomes significant, and hence accepting that $r_e$ could have a very small non-zero value. This is called the *classical electron radius.*

Regularization techniques can lead to what we call *anomalies.* An anomaly in Quantum Field Theory is when a certain symmetry exists in the classical action of a theory but is broken on the quantum level when the theory is regularized. This could be because the terms that regularize the divergent action do not obey a certain symmetry which the original action preserved.

Here we are specifically interested in the anomaly arising from the break-down of the conformal symmetry of CFT's. Because the variation of the action is proportional to the energy-momentum tensor, the conformal re-scaling is subsequently related to the trace of the energy-momentum tensor. The conformal anomaly then exists when the expectation value of the CFT energy-momentum tensor does not vanish,

$$\gamma^{ab}\langle \hat{T}_{ab}\rangle \neq 0. \tag{3.25}$$

The AdS/CFT duality provides a tool for calculating the conformal anomaly of a CFT using data from the gravitational theory, namely contractions of the Riemann tensor and their derivatives.

Conformal anomalies only arise for even-dimensional CFT's. The reasons for this are purely mathematical; studies of conformal structures lead to different results in odd and even dimensions. In the latter, the theory leads to logarithmic divergences that break conformal symmetry [42]. The theorems behind this are discussed in details in [43].

Talk of conformal invariance will probably remind the reader of a particular quantity in GR that we know to also be conformally-invariant: the Weyl tensor $C^a{}_{bcd}$. Because the Weyl tensor is conformally invariant, it is sometimes referred to as the *conformal tensor*; subsequently, conformal anomaly is sometimes referred to as the *Weyl anomaly.*

## 3.3 Divergences in AdS Spacetimes

### 3.3.1 Introduction

The general black hole solution in an asymptotically $\mathrm{AdS}_{n+1}$ spacetime admitting spherical symmetry is given by [44]:

$$ds^2 = -\left(1 - \frac{2m}{r^{n-2}} + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r^{n-2}} + \frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{n-1}, \tag{3.26}$$

where $n \geq 2$, and $m$ is some real constant.

Notice an extra factor of $r^2/l^2$ in the ansatz function in comparison to the asymptotically flat case. The different form of the metric in AdS spacetimes often leads to divergences, particularly in the calculation of the action and the mass. Since we expect solutions with only cylindrical symmetry to reduce to the spherically-symmetric case when certain parameters are set to specific values, we can expect divergences in solutions admitting only cylindrical symmetry as well.

The full action in $D = n + 1$ dimensions is the sum of the Einstein-Hilbert action and the Gibbons-Hawking action ([4]):

$$I = -\frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^{n+1}x\sqrt{-g}\left(R + \frac{n(n-1)}{\ell^2}\right) - \frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^n\sqrt{-h}K, \tag{3.27}$$

where $h_{ab}$ and $K_{ab}$ are again the induced metric on and extrinsic curvature of the boundary respectively.

$$\beta = 1/T. \tag{3.28}$$

As an example, let us look at the solution for a rotating black hole in $AdS_4$. This solution is often referred to as Kerr-$AdS_4$. All quantities calculated below will reduce to the spherically-symmetric Schwarzschild-$AdS_4$ solution if we set $a = 0$. The metric is given by ([9]):

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\ &+ \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\mathrm{d}t - \frac{(r^2+a^2)}{\Xi}\mathrm{d}\phi\right)^2, \end{aligned} \tag{3.29}$$

where

$$\Xi = 1 - \frac{a^2}{\ell^2}, \tag{3.30}$$
$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{3.31}$$

$$\Delta_r = (r^2 + a^2)\left(1 + \frac{r^2}{\ell^2}\right) - 2mr, \tag{3.32}$$

$$\Delta_\theta = 1 - \left(\frac{a^2}{l^2}\right)\cos^2\theta. \tag{3.33}$$

In the following parts of this section, our discussion will be based on results in [18].

As discussed in Appendix D, we make the coordinate transformation $t \to i\tau$ and and impose a period $\beta = 1/T$ on the imaginary time coordinate [5]. The inverse temperature can be found by direct computation to be

$$\beta = \frac{4\pi(r_+^2 + a^2)}{r_+(1 + 3r_+^2/\ell^2 + a^2/\ell^2 - a^2/r{+}^2)}. \tag{3.34}$$

We find the Einstein-Hilbert action to be given by

$$I_{\mathrm{EH}}/\beta = \frac{r^3}{2\ell^2\Xi}\Big|_{r\to+\infty} + \frac{a^2 r}{2\ell^2\Xi}\Big|_{r\to+\infty} - \frac{r_+\left(a^2 + r_+^2\right)}{2\ell^2\Xi} + \mathcal{O}\left(r^{-1}\right). \tag{3.35}$$

Evidently, the first two terms are divergent.

Next, we give the Gibbons-Hawking action:

$$I_{\mathrm{GH}}/\beta = -\frac{3r^3}{2\ell^2\Xi}\Big|_{r\to+\infty} + \left(-\frac{5a^2}{6\ell^2\Xi} - \frac{1}{\Xi}\right) r\Big|_{r\to+\infty} + \frac{3m}{2\Xi} + \mathcal{O}\left(r^{-1}\right). \tag{3.36}$$

The total action, which also diverges, is thus given by

$$I_{\mathrm{KAdS}_4} = -\frac{\beta r^3}{\ell^2}\Big|_{r\to+\infty} - \left(1 + \frac{a^2}{3\ell^2}\right)\beta r\Big|_{r\to+\infty} + \frac{3\ell^2 m - r_+\left(a^2 + r_+^2\right)}{2\ell^2}\beta. \tag{3.37}$$

In discussing the above results, we have omitted a series of tedious calculations. The reason we are not very concerned with showing the details at this point is that we will be discussing in great detail the exact calculations for the more involved case of a charged, rotating black hole in $\mathrm{AdS}_5$. Here we are simply interested in showing the above result, which obviously diverges as $r$ goes to infinity. Higher dimension solutions will include divergent terms in higher powers of $r$ as well (see ref. [45]).

As we mentioned, the energy also diverges. In Chapter 1 we saw two reasonable ways of defining the energy of the solution. The first is the Komar integral. It can be easily verified from Killing's equation that the timelike Killing vector for the solution in (3.29) can still be expressed by

$$\xi = \partial_t. \tag{3.38}$$

Using equation (1.103), we find the Komar mass to be

$$M_{\mathrm{K}} = \frac{r^3 + a^2 r + m\left(l^2 - 2a^2/\Xi\right)}{\ell^2 \Xi}\Big|_{r\to+\infty}, \tag{3.39}$$

which also diverges.

The second way we saw to define the energy is the Brown-York method in equation (1.130). Using the metric in (3.29), we get

$$\begin{aligned} M_{BY} =& \frac{r^3\left(3a^2 - 3\ell^2\right)}{3\ell^4 \Xi}\Big|_{r\to+\infty} + \frac{r\left(a^4 + 2a^2 l^2 - 3\ell^4\right)}{3\ell^4 \Xi}\Big|_{r\to+\infty} \\ &+ \frac{6\ell^4 m - 4a^2\ell^2 m}{3\ell^4 \Xi}. \end{aligned} \tag{3.40}$$

Again, this method also leads to a divergent mass.

Reference [46] shows that these action and mass divergences also occur in 3, 5 and 6 dimensional AdS spacetimes.

Discovering that a certain definition in physics diverges is nothing new. We encounter this in several cases in classical and quantum physics. This simply means that we need to alter our definitions in order to "regularize" these divergences. Ideally, we would like to find ways to regularize these AdS divergences which leave our asymptotically flat definitions unchanged. The reason is that the latter seem to work fine; they satisfy a first law of black hole thermodynamics, and under certain assumptions, reduce to familiar results from Newtonian mechanics, as we have seen for the mass and angular momentum in Chapter 1.

There are two regularization methods which are widely used methods in the literature: the background subtraction method and the counterterms subtraction method. In the next two subsections we will discuss these methods in detail, and will show how they solve the divergences in the Kerr-AdS$_4$ case. We will then use both methods to study the main metric of the thesis in the following chapters.

### 3.3.2 The Background Subtraction Method

The background subtraction method, as the name suggests, resolves the divergence problems by defining the action and isometric charges with respect to a background spacetime. For this to work, the original metric and the background metric must have the same boundary at infinity. In the asymptotically AdS four-dimensional Kerr solution (3.29) for example, it is convenient to chose the background as a rotating pure AdS spacetime. This can be done by setting $m = 0$ in the metric.

Applying this to the action in (3.37), we find the regularized action below

$$\begin{aligned} \tilde{I}_{\mathrm{KAdS}_4} &= I_{\mathrm{KAdS}_4} - I_{\mathrm{KAdS}_4}\Big|_{m=0} \\ &= \frac{3m}{2\Xi}\beta. \end{aligned} \tag{3.41}$$

The Komar mass can also be regularized using background subtraction. This leads to

$$\begin{aligned} M &= M_{\mathrm{K}} - M_{\mathrm{K}}\Big|_{m=0} \\ &= \frac{m}{\Xi}. \end{aligned}$$

This result was found by Hawking, Hunter and Taylor-Robinson [47]. However, it was noted in [6] that this mass does not satisfy the first law of thermodynamics. The reason is that the timelike Killing vector in the $\mathrm{AdS}_4$ case should actually be re-scaled to ([18])

$$\chi = \partial_t/\Xi. \tag{3.42}$$

We can calculate the mass again using this re-scaled Killing vector. This gives us

$$M = \frac{m}{\Xi^2}. \tag{3.43}$$

The background subtraction method, while widely used in the literature (especially before the rise in popularity of the counterterms subtraction method), has certain problems. First, by subtracting the values of the pure AdS spacetime from the spacetime containing the black hole, we eliminate any physics common between the two spacetimes. For instance, we will see in §3.4 that odd-dimensional AdS spacetimes have a finite vacuum energy. Evidently this cannot be calculated using the background subtraction method. This is also the case for the physical quantity corresponding to the conformal anomaly discussed in §3.2.3 [8].

Furthermore, while the background subtraction method requires a reference background, there are certain solutions where an appropriate background spacetime is ambiguous or unknown (for example the Taub-NUT-AdS and Taub-Bolt-AdS) [4].

### 3.3.3 The Counterterms Subtraction Method

A different regularization scheme can be found in [42], where it was shown that the action divergences in AdS spacetimes can be written in terms of local integrals of the metric of the boundary CFT $\gamma_{ab}$. This was inspired by the AdS/CFT correspondence. It was further elaborated on by Vijay Balasubramanian and Per Kraus in [38] to give these divergences in terms of the induced boundary metric $h_{ab}$.

The counterterms subtraction technique is based on the fact that one can then construct counterterms from these local integrals which can be added to the action to render it finite. The final non-divergent action is given by

$$I = I_{\mathrm{EH}} + I_{GH} + I_{\mathrm{ct}}, \tag{3.44}$$

with the latter taking the form

$$I_{\mathrm{ct}} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} \mathrm{d}^n \sqrt{-h} f(\ell, \mathcal{R}[h_{ab}], \mathcal{D}\mathcal{R}[h_{ab}]), \tag{3.45}$$

and depending only on the Ricci scalar of the boundary $\mathcal{R}$ and its derivative $\mathcal{D}_a\mathcal{R}$ [4].[1]

Particularly, the first couple of counterterms are given by ([4])

$$\boxed{I_{\mathrm{ct}} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} \mathrm{d}^n x \sqrt{-h} \left[\frac{n-1}{\ell} + \frac{\ell\mathcal{R}}{2(n-2)} + ...\right]}, \tag{3.46}$$

where the dots indicate the appearance of more terms in higher dimensions. The above terms are enough to cancel divergences for $n \leq 4$, which is what we will be interested in.

We will now apply the counterterms subtraction technique to the Kerr-AdS$_4$ metric (3.29) to illustrate the use of this technique. The divergent parts of the action are still given by (3.35) and (3.36). Let us call $\alpha$ the integrand of the integral (3.46). It is given by

[1] Here $\mathcal{D}_a$ is the covariant derivative defined on the boundary manifold.

$$
\begin{aligned}
\alpha = &-\frac{\beta}{4\sqrt{2}l^2\Xi\Big(a^2\cos(2\theta)+a^2+2r^2\Big)^{5/2}}\Big(\sin\theta\times \\
&\sqrt{a^2\left(l^2+r^2\right)+r\left(l^2(r-2m)+r^3\right)}\big(a^6\cos(6\theta)+10a^6+8a^4r^2 \\
&+32a^2l^2mr-16a^2l^2r^2-48a^2r^4+\left(6a^6+8a^4r^2\right)\cos(4\theta)+a^2\cos(2\theta) \\
&\left(15a^4+16a^2r^2-16r\left(l^2(r-2m)+r^3\right)\right)-32l^2r^4-64r^6\big)\Big).
\end{aligned}
\tag{3.47}
$$

Series-expanding around infinity, we get

$$
\alpha/\beta = \frac{2r\sin\theta\left(l^2-a^2\cos(2\theta)\right)}{l^2\Xi}+\frac{2r^3\sin\theta}{l^2\Xi}-\frac{2(m\sin\theta)}{\Xi}+\mathcal{O}(r^{-1}). \tag{3.48}
$$

The counterterms action is then found by integration:

$$
I_{\mathrm{ct}}/\beta = \left[\frac{r^3}{l^2\Xi}+\left(\frac{a^2}{3l^2\Xi}+\frac{1}{\Xi}\right)r-\frac{m}{\Xi}\right]\Bigg|_{r\to+\infty}. \tag{3.49}
$$

Combining (3.35), (3.36) and (3.49), the final action is given by

$$
I_{\mathrm{KAdS}_4} = \frac{2\pi r_+\left(a^2+r_+^2\right)\left(a^2r_+-m\ell^2+r_+^3\right)}{\left[a^2\left(l^2-r_+^2\right)-r_+^2\left(l^2+3r_+^2\right)\right]\Xi}, \tag{3.50}
$$

This results matches the one found in [9]. We see that the final action is in fact finite. We can now use it in equation (1.130) to calculate the Brow-York energy. The quasi-local energy momentum tensor will now include two additional terms coming from the new counterterm action in (3.46),

$$
T^{ab}_{\mathrm{ct}} = -\frac{2}{\sqrt{-h}}\frac{\delta I_{\mathrm{ct}}}{\delta h_{ab}}. \tag{3.51}
$$

We have already seen in Appendix D that

$$
\frac{\delta\sqrt{-h}}{\delta h_{ab}} = -\frac{1}{2}\sqrt{-h}h^{ab}, \tag{3.52}
$$

so the first additional term in the stress tensor is

$$\frac{(n-1)}{\ell}h^{ab}.$$

We have also seen the variation of the Ricci scalar will yield the Einstein tensor. We denote the Einstein tensor of the boundary by $\mathcal{G}_{ab}$. The full quasi-local stress tensor is therefore given by

$$T^{ab} = \frac{1}{8\pi}\left[K^{ab} - h^{ab}K + \frac{(n-1)}{\ell}h^{ab} - \frac{\ell\mathcal{G}^{ab}}{(n-2)}\right]. \tag{3.53}$$

The Killing vector will be that in (3.42). The stress tensor now has four terms and the mass can thus be divided into four terms as well. The contribution of the first term gives

$$M_1 = \left.\left[\frac{m\left(5a^2-3\ell^2\right)}{6\Xi^2\left(a^2-\ell^2\right)} + \frac{a^2 r}{2\ell^2\Xi^2} + \frac{r^3}{2\ell^2\Xi^2}\right]\right|_{r\to+\infty}. \tag{3.54}$$

The second term gives

$$M_2 = \left.\left[\frac{a^2m-3\ell^2m}{2\Xi^2\left(a^2-\ell^2\right)} + \frac{r\left(-5a^2-6\ell^2\right)}{6\ell^2\Xi^2} - \frac{3r^3}{2\ell^2\Xi^2}\right]\right|_{r\to+\infty}. \tag{3.55}$$

Lastly, the contribution of the counterterms action gives

$$M_{\text{ct}} = \left.\left[-\frac{m\left(a^2-3\ell^2\right)}{3\Xi^2\left(a^2-\ell^2\right)} + r\left(\frac{a^2}{3\ell^2\Xi^2} + \frac{1}{\Xi^2}\right) + \frac{r^3}{\ell^2\Xi^2}\right]\right|_{r\to+\infty}. \tag{3.56}$$

By careful inspection, it is easy to see that the divergences in the three terms cancel perfectly. Combining the above equations, we arrive at the following expression for the energy:

$$M = \frac{m}{\Xi^2}. \tag{3.57}$$

Notice that when applying the counterterms subtraction technique, we did not specify a particular background, contrary to the previous technique. This allows us for example to calculate a finite non-zero energy for the pure AdS space, which would not be possible using the background subtraction method since this energy would depend on the choice of the background. Particularly, since it makes most sense to take the background to be the AdS spacetime itself, then the energy would always be zero. We will calculate the energy of the pure AdS spacetime using the counterterms subtraction technique in §3.4.

### 3.3.4 Discussion of the Thermodynamics in Asymptotically $AdS_4$ Spacetimes

Having discussed how the divergence in energy can be cancelled, it is logical to ask what impact the regularization schemes we discussed have on the first law. In fact, it is logical to wonder whether the first law even holds in AdS spacetimes. This turns out not to be somewhat of a complicated issue.

Let us start by looking at the first law in Kerr-$AdS_4$. Conveniently, the masses calculated in (3.43) and (3.57) are the same. We now need to calculate the angular momentum. The Komar integral in (1.104) directly evaluates to

$$J_\phi = \frac{am}{\Xi^2}, \tag{3.58}$$

which does not diverge. The same result is found if we instead use the Brown-York formula in (1.131). These results match those in [47] and [6]. So in conclusion, there is no ambiguity with the definition of the angular momentum either. Obviously, there is no ambiguity with the temperature and entropy as well, which are given by ([47])

$$T = \frac{r_+(1 + a^2\ell^{-2} + 3r_+^2\ell^{-2} - a^2 r_+^{-2})}{2(r_+^2 + a^2)}, \tag{3.59}$$

and

$$S = \frac{\pi(r_+^2 + a^2)}{\Xi}. \tag{3.60}$$

The only quantity which seems a little ambiguous is the angular velocity. Recall that in Chapter 1 we discussed that the angular velocity can be chosen as $\Omega_H$ or $\Omega_H - \Omega_\infty$. In the asymptotically-flat case this was not an issue since $\Omega_\infty$ vanished. However, direct evaluation of (1.61) at $r \to \infty$ gives a non-vanishing expression,

$$\Omega_\infty = \frac{a}{\ell^2}. \tag{3.61}$$

It was shown in [6] that only the quantity $\Omega_H - \Omega_\infty$ satisfies the first law when the correct timelike Killing vector (3.42) is used. It is given explicitly by

$$\Omega = \frac{a(1 + r_+^2 l^{-2})}{r_+^2 + a^2}. \tag{3.62}$$

Furthermore, it was shown in [18] that the same outcome holds for charged rotating black holes in $\mathrm{AdS}_4$ as well. So from now on, we never say angular velocity we mean the quantity $\Omega = \Omega_H - \Omega_\infty$.

Two more definitions [48, 49] for the mass were also considered in [6], which yield the same expression as (3.43), and hence they also satisfy the first law in four-dimensional AdS spacetimes. However, it was shown that neither of the four expressions satisfy the first law in dimensions higher than 4. It was hence suggested in [6] that to have an expression for the mass that satisfies the first law, one should integrate the right-hand side of the first law. We find this method of defining the mass to be unfavourable for reasons that we will discuss in the next chapter. We will also see in the next chapter that the mass calculated using the counterterms subtraction method can be made to satisfy a more rigorous version of the first law.

## 3.4 Casimir Energy and the AdS/CFT Correspondence

You should recall that in Quantum Mechanics the value of the vacuum energy is non-zero due to the Uncertainty Principle which allows for particle-antiparticle pairs to be created and annihilated. This energy is not some fictitious theoretical concept without evidence. It can actually leads to measurable quantities, most famously the *Casimir effect*. In Quantum Field Theory, the vacuum expectation value of the electromagnetic energy operator is modified by the presence of two conductive plates separated by a dielectric. The modification of this expectation value leads to a force between the two plates known as the *Casimir force*.

This Casimir effect can be visualized as follows: in vacuum, particle-antiparticle pairs are constantly created and annihilated. Consequently, the vacuum at any one time is filled with a number of particles (and antiparticles). If we bring in two plates separated by a certain distance, then the number of particles in the region between the two plates is (much) smaller than the number of particles surrounding the two plates. This creates a pressure on the two plates which tries to push them closer. Due to this effect, the vacuum energy in Quantum Field Theory is often referred to as the *Casimir energy*.

An $\mathrm{AdS}_5 \times S^5$ gravitational theory resulting from String Theory is expected to be dual to a class of conformal field theories called $\mathrm{SU}(N)$ super-Yang-Mills. Without getting into too much details that are beyond our scope, we can simply say that Yang-Mills theories are a special class of gauge theories which have non-Abelian gauge symmetry groups. The prefix "super" means that these theories have an additional type of symmetry called *supersymmetry*. As the reader may know, supersymmetry is a presumed symmetry between bosons and fermions. This means that for each fermion there is a corresponding boson which has the same intrinsic charges as the fermion and only differs in spin. Likewise, for every boson there is also a corresponding fermion with equivalent charges but different spin. As we know from the AdS/CFT correspondence, the super-Yang-Mills theory is of course defined on the AdS$_5$ boundary and has topology $S^3 \times \mathbb{R}$. When such theory is defined on a sphere of radius $R$, it has a Casimir energy given by ([38]):

$$E_{\text{Casimir}} = \frac{3(N^2-1)}{16R}. \tag{3.63}$$

Since we are interested in the large-$N$ limit, this can be approximated to

$$E_{\text{Casimir}} = \frac{3N^2}{16R}. \tag{3.64}$$

The radius of the sphere on which the theory resides is of course the AdS radius $\ell$. We would now like to see if this quantity corresponds to some quantity in the bulk gravitational theory. To do so, let us write the Schwarzschild-AdS$_5$ [38]:

$$ds^2 = -\left[1 - \left(\frac{2mG}{r}\right)^2 + \frac{r^2}{\ell^2}\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left[1-\left(\frac{2mG}{r}\right)^2 + \frac{r^2}{\ell^2}\right]} + r^2\mathrm{d}\Omega_3^2. \tag{3.65}$$

We have reinstated the gravitational coefficient $G$ because it has a connection to the AdS/CFT correspondence which we will make use of. Since we are looking for some quantity that would be dual to an energy, let us use the counterterms method to calculate the energy of the black hole solution. The contribution to mass arising from the first term of the surface action gives

$$M_1 = \frac{1}{4G}\pi\left(g^2r^4 + 2m\right). \tag{3.66}$$

That of the second term is

$$M_2 = \frac{\pi m}{G} - \frac{1}{4G}\pi r^2\left(4g^2r^2 + 3\right). \tag{3.67}$$

Finally, the contributions from the counterterms give

$$M_{\text{ct}} = \frac{3\pi\left(8g^4r^4 - 8g^2\left(m - r^2\right) + 1\right)}{32Gg^2}. \tag{3.68}$$

Putting those together, we find the final value of the mass to be

$$M = \frac{3\pi}{32Gg^2} + \frac{3\pi m}{4G}, \tag{3.69}$$

in complete agreement with [38]. The energy calculated in (3.69) is the energy of the complete Schwarzschild-$\mathrm{AdS}_5$ solution. We should be able to retrieve the pure $\mathrm{AdS}_5$ spacetime by setting $m = 0$, since this would transform the metric in (3.65) to the pure $\mathrm{AdS}_5$ metric. Interestingly, this leaves us with a non-null energy in (3.69), given by

$$M_{\mathrm{BG}} = \frac{3\pi}{32Gg^2}. \tag{3.70}$$

The quantity $M_{\mathrm{BG}}$ is hence the background energy of the $\mathrm{AdS}_5$ spacetime. It is understood in the framework of the AdS/CFT correspondence that the quantity $\pi/(2Gg^3)$ in the gauge theory corresponds to the quantity $N^2$ in the dual field theory [38]. Plugging this into the Casimir energy, we can see that the Casimir energy of the dual field theory equals the background energy of the bulk theory in the large-$N$ limit,

$$M_{\mathrm{Casimir}} = M_{\mathrm{BG}}. \tag{3.71}$$

This relation was shown to hold in rotating $\mathrm{AdS}_5$ solutions as well in [9]. In the following chapter we will investigate this relation in the case of charged rotating black holes.

You might notice that in equation (3.57) there was no background energy for the 4-dimensional rotating black hole. In fact, like conformal anomalies, Casimir energies only in even dimensional conformal field theories, and hence background energies only appear in odd dimensional gravitational solutions.

Explicit calculations for the Casimir energy for neutral black holes in 3 dimensions can be found in [38], and in 5 and 7 dimensions in [45].

# Chapter 4

# Thermodynamics of Charged Rotating Black Holes in $\mathrm{AdS}_5$

## 4.1 Presentation and Discussion of the Solution

We are now ready to tackle a generic solution for charged rotating black holes in five-dimensional anti-de Sitter spacetime. The solution was provided in [10] by

$$
\begin{aligned}
\mathrm{d}s^2 = &-\frac{\Delta_\theta[(1+g^2r^2)\rho^2\mathrm{d}t + 2q\nu]dt}{\Xi_a\Xi_b\rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4}\left(\frac{\Delta_\theta \mathrm{d}t}{\Xi_a\Xi_b} - \omega\right)^2 \\
&+ \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2\mathrm{d}\theta^2}{\Delta_\theta} + \frac{r^2+a^2}{\Xi_a}\sin^2\theta\,\mathrm{d}\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta\,\mathrm{d}\psi^2,
\end{aligned}
\tag{4.1}
$$

where

$$g = 1/\ell, \tag{4.2}$$

$$\nu = b\sin^2\theta\,\mathrm{d}\phi + a\cos^2\theta\,\mathrm{d}\psi, \tag{4.3}$$

$$\omega = a\sin^2\theta\frac{\mathrm{d}\phi}{\Xi_a} + b\cos^2\theta\frac{\mathrm{d}\psi}{\Xi_b}, \tag{4.4}$$

$$\Delta_\theta = 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta, \tag{4.5}$$

$$\Delta_r = \frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2) + q^2 + 2abq}{r^2} - 2m, \tag{4.6}$$

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \tag{4.7}$$

$$\Xi_a = 1 - a^2 g^2, \tag{4.8}$$

$$\Xi_b = 1 - b^2 g^2, \tag{4.9}$$

$$f = 2m\rho^2 - q^2 + 2abqg^2\rho^2. \tag{4.10}$$

Note that the constant $g = 1/\ell$ is not the determinant of the metric. Another important remark is that in four spatial dimensions, the black hole has *two* possible rotation axes. To see this, first start with a point in one spatial dimension. We obviously have no rotation axes in one dimension since we cannot define a rotation plane. In two dimensions the space itself can be a rotation plane, so we can define one rotation axis. Likewise, in three dimensions we can always project any rotation on a single plane (this is in fact due to *Euler's rotation theorem*). In four spatial dimensions it is obvious that we can have two orthogonal rotation planes and hence we have two possible independent rotation axes.

The coordinate system in (4.1) is evidently $(t, r, \theta, \phi, \psi)$. It is easy to understand that the range of the last three coordinates is set so they can cover a three-sphere. This is similar to a hyperspherical coordinate system and we know that the corresponding range for $\theta$ is between 0 and $\pi/2$, while that for $\phi$ and $\psi$ is between 0 and $2\pi$. The two rotation directions are specified using the $\phi$ and $\psi$ coordinates. The angular velocities are denoted $\Omega_a$ (in the $\phi$-direction) and $\Omega_b$ (in the $\psi$-direction). They are given in [10] by

$$\begin{aligned} \Omega_a &= \frac{a\left(r_+^2 + b^2\right)\left(1 + g^2 r_+^2\right) + bq}{\left(r_+^2 + a^2\right)\left(r_+^2 + b^2\right) + abq}, \\ \Omega_b &= \frac{b\left(r_+^2 + a^2\right)\left(1 + g^2 r_+^2\right) + aq}{\left(r_+^2 + a^2\right)\left(r_+^2 + b^2\right) + abq}. \end{aligned} \tag{4.11}$$

The electromagnetic four-potential is also given in [10] by

$$A = \frac{\sqrt{3}q}{\rho^2}\left(\frac{\Delta_\theta \mathrm{d}t}{\Xi_a \Xi_b} - \omega\right). \tag{4.12}$$

The horizon Killing vector is

$$\xi = \partial_t + \Omega_a \partial_\phi + \Omega_b \partial_\psi. \tag{4.13}$$

The temperature, surface gravity and electric charge are given by

$$T = \frac{r_+^4 \left[g^2 \left(a^2 + b^2 + 2r_+^2\right) + 1\right] - (ab+q)^2}{2\pi r_+ \left[\left(a^2 + r_+^2\right)\left(b^2 + r_+^2\right) + abq\right]}, \tag{4.14}$$

$$S = \frac{\pi^2 \left[\left(r_+^2 + a^2\right)\left(r_+^2 + b^2\right) + abq\right]}{2\Xi_a \Xi_b r_+}, \tag{4.15}$$

$$Q = \frac{\sqrt{3}\pi q}{4\Xi_a \Xi_b}. \tag{4.16}$$

Note that we have an axial symmetry in $\phi$ and $\psi$. The Killing vectors are obviously given by $\eta_a := \partial_\phi$ and $\eta_b := \partial_\psi$. The angular momenta were subsequently calculated in [10] using the Komar integral, which in five dimensions takes the form

$$J_{a/b} = \frac{1}{16\pi} \int_{S^3} \star \mathrm{d}\eta_{a/b}, \tag{4.17}$$

yielding

$$J_a = \frac{\pi \left[2am + qb\left(1 + a^2 g^2\right)\right]}{4\Xi_a^2 \Xi_b}, \tag{4.18}$$

$$J_b = \frac{\pi \left[2bm + qa\left(1 + b^2 g^2\right)\right]}{4\Xi_b^2 \Xi_a}. \tag{4.19}$$

We see that the angular momenta remain finite just as in the Kerr-AdS$_4$ case.

Finally, an expression for the mass was proposed in [10] by integration of the first law. This mass is given by

$$M_0 = \frac{\pi m \left(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b\right) + 2\pi a b g^2 q \left(\Xi_a + \Xi_b\right)}{4\Xi_a^2 \Xi_b^2}. \tag{4.20}$$

The mass was denoted in [10] by $M$ but we will use here the notation $M_0$ to distinguish this quantity from the mass expression that we have calculated. Note that if we reduce this solution to a Schwarzschild black hole (i.e. we set $q = 0$, $a = b = 0$), the mass is proportional to $m$,

$$M_0\Big|_{q=0, a=b=0} = \frac{3\pi m}{4}.$$

This means that the constant quantity $m$ can be interpreted as our usual "mass parameter". It is a peculiar feature of this solution that when the mass parameter goes to 0, the black hole's mass does not vanish due to contributions from the charge given by

$$M_0\Big|_{m=0} = \frac{\pi abg^2 q(\Xi_a + \Xi_b)}{2\Xi_a^2\Xi_b^2}.$$

For instance, the charged black hole solution in five-dimensional anti-de Sitter spacetime was given in [50] by

$$M = \frac{(n-1)\Omega_{n-1}}{16\pi}m,$$

where $\Omega_{n-1}$ is the volume of the unit $(n-1)$-sphere. Here not only does the mass vanish when $m$ goes to 0, but it is also totally independent of the parameter $q$, just as in the Kerr-Newman case. It is worth noting that this interesting feature will remain the same when we calculate a different expression for the mass using the counterterms subtraction methods.

The rest of the chapter is organized as follows: in §4.2 we will check to see if the solution satisfies the usual thermodynamical relations: the first law and the relations between the Gibbs potential and the extensive quantities of the first law. In §4.3 we will use the counterterms subtraction method to calculate the action, mass and angular momenta of the solution. We will then look at features from the AdS/CFT correspondence: we will calculate the conformal anomaly from the gravitational theory and the conformal field theory and check if the two expressions match. We will further calculate the background energy of the gravitational theory then calculate the vacuum energy of the dual field theory on the AdS boundary and compare the two expressions as well. Lastly, we will also address the fate of the first law in this solution.

## 4.2 Verification of the Thermodynamical Relations of the Original Solution

We start by verifying the first law, which in this case is given by

$$\mathrm{d}M = T\mathrm{d}S + \Omega_a \mathrm{d}J_a + \Omega_b \mathrm{d}J_b + \Phi \mathrm{d}Q. \tag{4.21}$$

Since the electric potential was not given in [10], we will calculate it using the expression in (1.64). To perform the calculation we need to evaluate the expression $\xi^a A_a$ at infinity and then subtracted from its value evaluated at the horizon. Direct evaluation of the first gives

$$\lim_{r\to\infty} \xi^a A_a = 0. \tag{4.22}$$

Direct evaluation of the second part gives

$$\xi^a A_a|_{r_+} = -\frac{\sqrt{3}q\left(\cos^2\theta\Omega_b\Xi_a b + \sin^2\theta\Omega_a\Xi_b a - \Delta_\theta\right)}{\left(a^2\cos^2\theta - b^2\cos^2\theta + b^2 + r_+^2\right)\Xi_a\Xi_b}. \tag{4.23}$$

We can now use the values of $\Omega_a$ and $\Omega_b$, and simplify the above expression using Maple. This yields

$$\boxed{\Phi = \frac{\sqrt{3}qr_+^2}{\left(a^2+r_+^2\right)\left(b^2+r_+^2\right)+abq}}. \tag{4.24}$$

To verify the first law, we need to verify four equations of the form

$$\frac{\partial M}{\partial\alpha}\mathrm{d}\alpha = T\frac{\partial S}{\partial\alpha}\mathrm{d}\alpha + \Phi\frac{\partial Q}{\partial\alpha}\mathrm{d}\alpha + \Omega_a\frac{\partial J_a}{\partial\alpha}\mathrm{d}\alpha + \Omega_b\frac{\partial J_b}{\partial\alpha}\mathrm{d}\alpha, \tag{4.25}$$

with $\alpha \in \{r_+, q, a, b\}$ since the mass is a function of these variables.

**Differentiations with respect to $r_+$:**

Direct evaluation of the mass differentiation gives

$$\frac{\partial M}{\partial r_+} = \frac{\pi(\Xi_a(\Xi_b-2)-2\Xi_b)\left(a^2\left(b^2-g^2r_+^4\right)+2abq+r_+^4\left(-\left(b^2g^2+2g^2r_+^2+1\right)\right)+q^2\right)}{4r_+^3\Xi_a^2\Xi_b^2}. \tag{4.26}$$

And the other differentiations are given by

$$\frac{\partial S}{\partial r_+} = \frac{\pi^2 \left(a^2 \left(r_+^2 - b^2\right) - abq + r_+^2 \left(b^2 + 3r_+^2\right)\right)}{2r_+^2 \Xi_a \Xi_b}, \tag{4.27}$$

$$\frac{\partial J_a}{\partial r_+} = \frac{\pi a \left(a^2 \left(g^2 r_+^4 - b^2\right) - 2abq + r_+^4 \left(b^2 g^2 + 2g^2 r_+^2 + 1\right) - q^2\right)}{2r_+^3 \Xi_a^2 \Xi_b}, \tag{4.28}$$

$$\frac{\partial J_b}{\partial r_+} = \frac{\pi b \left(a^2 \left(g^2 r_+^4 - b^2\right) - 2abq + r_+^4 \left(b^2 g^2 + 2g^2 r_+^2 + 1\right) - q^2\right)}{2r_+^3 \Xi_a \Xi_b^2}, \tag{4.29}$$

$$\frac{\partial Q}{\partial r_+} = 0. \tag{4.30}$$

Using these expressions to evaluate the right-hand side of the first law directly yields

$$\frac{\pi(\Xi_a(\Xi_b - 2) - 2\Xi_b) \left(a^2 \left(b^2 - g^2 r^4\right) + 2abq + r^4 \left(-\left(b^2 g^2 + 2g^2 r^2 + 1\right)\right) + q^2\right)}{4r^3 \Xi_a^2 \Xi_b^2},$$

which we verify, using Mathematica, is the same value as that given in (4.26). In general, the expressions of the left-hand and right-hand sides in this section are quite complicated and we simply use Mathematica to verify that they do equate to each other.

**Differentiations with respect to $q$**

The differentiation on the left-hand side gives

$$\frac{\partial M}{\partial q} = \frac{\pi \left(2\Xi_b \left(a \left(bg^2 r_+^2 + b\right) + q\right) + \Xi_a \left(2 \left(abg^2 r_+^2 + ab + q\right) - \Xi_b (ab + q)\right)\right)}{4r_+^2 \Xi_a^2 \Xi_b^2}. \tag{4.31}$$

The right-hand side differentiations are given by

$$\frac{\partial S}{\partial q} = \frac{\pi^2 ab}{2r_+ \Xi_a \Xi_b}, \tag{4.32}$$

$$\frac{\partial J_a}{\partial q} = \frac{\pi \left(a^2 b g^2 + \frac{2a(ab+q)}{r_+^2} + b\right)}{4\Xi_a^2 \Xi_b}, \tag{4.33}$$

$$\frac{\partial J_b}{\partial q} = \frac{\pi \left(ab^2 g^2 + \frac{2b(ab+q)}{r_+^2} + a\right)}{4\Xi_a \Xi_b^2}, \tag{4.34}$$

$$\frac{\partial Q}{\partial q} = \frac{\sqrt{3}\pi}{4\Xi_a \Xi_b}. \tag{4.35}$$

These expressions also lead to a right-hand side that cancels the left-hand side.

**Differentiations with respect to $a$**

Since all equations are symmetric in $a$ and $b$, we will only discuss results for the differentiation with respect to $a$.

The left-hand side gives

$$
\begin{aligned}
\frac{\partial M}{\partial a} =& \frac{1}{4r_+^2\left(a^2g^2-1\right)^3\left(b^2g^2-1\right)^2}\pi\Big(a^4bg^4q\left(b^2g^2+2g^2r_+^2+1\right) \\
&+a^3g^2\Big(b^4g^2\left(g^4r_+^4+4g^2r_+^2+3\right) \\
&+b^2\left(g^6r_+^6+g^4\left(q^2+5r_+^4\right)+3g^2r_+^2-1\right)+g^4r_+^6+g^2q^2-r_+^2\Big) \\
&+6a^2bg^2q\left(b^2g^2-1\right)\left(g^2r_+^2+1\right)+a\Big(b^4\left(3g^6r_+^4+4g^4r_+^2+g^2\right) \\
&+b^2\left(3g^6r_+^6+g^4\left(3q^2-r_+^4\right)-7g^2r_+^2-3\right)-5g^4r_+^6-g^2\left(5q^2+8r_+^4\right)-3r_+^2\Big) \\
&+bq\left(b^2\left(2g^4r_+^2+g^2\right)-4g^2r_+^2-3\right)\Big).
\end{aligned} \tag{4.36}
$$

And the right-hand side differentiations are given by

$$
\begin{aligned}
\frac{\partial J_a}{\partial a} =& \frac{\pi}{4\Xi_a^3\Xi_b r_+^2}\Big(a^4g^2\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+2a^3bg^2q\left(g^2r_+^2+2\right) \\
&+3a^2\left(\left(bg^2r_+^2+b\right)^2+g^4r_+^6+g^2\left(q^2+2r_+^4\right)+r_+^2\right)+2abq\left(3g^2r_+^2+2\right) \\
&+r_+^2\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+q^2\Big),
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
\frac{\partial J_b}{\partial a} =& \frac{\pi}{4r_+^2\Xi_a^2\Xi_b^2}\Big(a^2g^2q\left(b^2\left(g^2r_+^2+2\right)+r_+^2\right)+2ab\Big(\left(bg^2r_+^2+b\right)^2+g^4r_+^6+g^2\left(q^2+2r_+^4\right) \\
&+r_+^2\Big)+q\left(b^2\left(g^2r_+^2+2\right)+r_+^2\right)\Big),
\end{aligned} \tag{4.38}
$$

$$
\frac{\partial S}{\partial a} = -\frac{\pi^2\left(a^2bg^2q+2a\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+bq\right)}{2r_+\Xi_a^2\Xi_b}, \tag{4.39}
$$

$$
\frac{\partial Q}{\partial a} = \frac{\sqrt{3}\pi a g^2 q}{2\Xi_a^2\Xi_b}. \tag{4.40}
$$

The right-hand side here automatically equates to the left-hand side. We conclude that the first law (4.21) is indeed verified.

We would now like to verify the thermodynamic relations (2.42) using the Gibbs potential

$$G(T, \Omega_a, \Omega_b, \Phi) = M - TS - \Omega_a J_a - \Omega_b J_b - \Phi Q. \tag{4.41}$$

Looking at the first relation,

$$S = -\left(\frac{\partial G}{\partial T}\right)_{\Omega_a, \Omega_b, \Phi},$$

we can see that it is difficult to evaluate the variations explicitly with respect to the temperature. Instead, we will make use of the chain rule. Given a multivariable function $f : \mathbb{R}^n \ni (x_1, x_2, ...x_n) \to \mathbb{R}$ and a functional $g[f]$, we have the relation

$$\frac{\partial g}{\partial x_i} = \frac{\partial g}{\partial f}\frac{\partial f}{\partial x_i}, \quad \forall\, i \in \{1, 2, ...n\}. \tag{4.42}$$

So the variation of $g$ with respect to $f$ can simply be written as

$$\frac{\partial g}{\partial f} = \frac{\partial g}{\partial x_i} \cdot \left(\frac{\partial f}{\partial x_i}\right)^{-1}. \tag{4.43}$$

We may now choose $f = T, g = G$ and $x_i = r_+$. The variation with respect to $T$ can be expressed as

$$\frac{\partial G}{\partial T} = \frac{\partial G}{\partial r_+}\left(\frac{\partial T}{\partial r_+}\right)^{-1}. \tag{4.44}$$

With this, we directly verify that

$$\begin{aligned} -\left(\frac{\partial G}{\partial T}\right)_{\Omega_a, \Omega_b, \Phi} &= \frac{\pi^2\left[\left(a^2 + r_+^2\right)\left(b^2 + r_+^2\right) + abq\right]}{2r_+ \Xi_a \Xi_b} && (4.45) \\ &= S. && (4.46) \end{aligned}$$

Continuing with the rest of the relations, we have

$$
\begin{aligned}
-\left(\frac{\partial G}{\partial \Omega_a}\right)_{T,\Phi,\Omega_b} &= -\frac{\partial G}{\partial r_+}\left(\frac{\partial \Omega_a}{\partial r_+}\right)^{-1}_{T,\Phi,\Omega_b} \\
&= \frac{\pi\left(a^3\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+a^2bq\left(g^2r_+^2+2\right)\right)}{4r_+^2\Xi_a^2\Xi_b} \\
&\quad + \frac{\pi\left(a\left(r_+^2\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+q^2\right)+bqr_+^2\right)}{4r_+^2\Xi_a^2\Xi_b} \\
&= J_a. \qquad (4.47)
\end{aligned}
$$

Likewise, we get

$$
\begin{aligned}
-\left(\frac{\partial G}{\partial \Omega_b}\right)_{T,\Phi,\Omega_a} &= -\frac{\partial G}{\partial r_+}\left(\frac{\partial \Omega_b}{\partial r_+}\right)^{-1}_{T,\Phi,\Omega_a} \\
&= \frac{\pi\left(a^2b\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+aq\left(b^2\left(g^2r_+^2+2\right)+r_+^2\right)\right)}{4r_+^2\Xi_a\Xi_b^2} \qquad (4.48) \\
&\quad + \frac{\pi\left(b\left(r_+^2\left(b^2+r_+^2\right)\left(g^2r_+^2+1\right)+q^2\right)\right)}{4r_+^2\Xi_a\Xi_b^2} \qquad (4.49) \\
&= J_b. \qquad (4.50)
\end{aligned}
$$

And lastly,

$$
\begin{aligned}
-\left(\frac{\partial G}{\partial \Phi}\right)_{T,\Omega_a,\Omega_b} &= -\frac{\partial G}{\partial r_+}\left(\frac{\partial \Phi}{\partial r_+}\right)^{-1}_{T,\Omega_a,\Omega_b} \qquad (4.51) \\
&= \frac{\sqrt{3}\pi q}{4\Xi_a\Xi_b} \qquad (4.52) \\
&= Q. \qquad (4.53)
\end{aligned}
$$

## 4.3 Calculations with the Counterterms Method

### 4.3.1 Motivation, Action and Mass Calculations

Recall that in reference [6], it is discussed that for black holes in anti-de Sitter spacetimes in more than 4 dimensions, the energy calculated using the counterterms method does not satisfy the first law in the form of (2.23). It is argued that, in order to obtain an expression for the black hole energy which satisfies (2.23) in $D \geq 5$ dimensions, one should integrate the right-hand side of (2.23). Based on this, the authors presented the expression for the mass in (4.20).

Admittedly, this action *does* satisfy the first law in the form (4.21). However, we believe that there is a number of problems with this procedure. First, in any solution, the validity of the first law should be studied and, hopefully, proven, whereas this procedure starts by assuming this validity exists. By doing so, it prevents us from verifying if a first law does exist for a particular black hole solution. Second, we would like our definition for energy to be related with the time-translational isometry of the metric. This is done by relating our definition for energy with a timelike Killing vector. However, there is no apparent procedure for relating the energy in equation (4.20) to a timelike Killing vector. Third, by not using the counterterms method, we are unable to recover a vacuum energy that we can compare to the Casimir energy of the dual conformal field theory on the boundary. By not using the counterterms method to regularize the action (and therefore the quasi-local stress tensor), we are also unable to calculate the conformal anomaly of the CFT from the gravity side.

An alternative approach to the problem of the first law was presented by Skenderis and Papadimitriou in reference [7] for a neutral rotating black hole. Here the authors find the first law by using a variational approach. This of course is a more rigorous direction than simply assuming that the form of the first law in (2.23), which was derived in a different solution and dimension, should remain the same in all cases. The authors find a modified first law from this variational approach given by

$$\mathrm{d}M = \delta_\sigma M + T\mathrm{d}S + \Omega_a \mathrm{d}J_a + \Omega_b \mathrm{d}J_b + \Phi \mathrm{d}Q, \tag{4.54}$$

where the first term on the left-hand side was found to be equal to the differentiation of the Casimir energy,

$$\delta_\sigma M = \delta M_{\text{Casimir}}. \tag{4.55}$$

Therefore, the form (2.24) of the first law holds well if the Casimir energy vanishes. This is why in the Kerr-AdS$_4$ case our counterterms mass did verify this form of the first law: in four dimensions there is no dual Casimir energy.

The results in reference [7] were derived for theories with no Chern-Simons terms, and it is interesting to see if they hold in such cases. The result (4.54) was also checked only for the non-charged case and we will aim to verify it in the charged black hole solution that we are interested in.

We start by calculating the action. Evidently, unless the action is regularized correctly, there is no sense in building an energy-momentum tensor from it and using it in the Brown-York quasi-local energy expression to calculate the mass of the black hole.

The first part we will look at in the action is the Maxwell term. Recall that in §3.3.3 we claimed that all action divergences can be written in terms of the Ricci scalar at the boundary and its covariant derivative. We therefore expect a term like the Maxwell action not to present any divergences at all. In fact, we verify right away that the integrand of the Maxwell action is of the order $\mathcal{O}(r^{-3})$ as $r$ goes to infinity. However, we were not able to evaluate the Maxwell action since the integrand $\sqrt{-g}\,F^{ab}F_{ab}$ turned out to be very complicated. In fact, several trials over Maple and Mathematica, using different coordinate systems, were all fruitless. While having the full action has its uses, we will not be needing an explicit expression for the Maxwell action in the following discussion.

Direct evaluation of the Chern-Simons action yields 0,

$$\begin{aligned} I_{\text{CS}} &= -\frac{1}{16\pi}\frac{1}{3\sqrt{3}}\int_{\mathcal{M}} F\wedge F\wedge A \\ &= 0. \end{aligned} \tag{4.56}$$

This is also the case for the Einstein-Maxwell-Chern-Simons solution without rotation in [50].

The Einstein-Hilbert action is found to be

$$I_{\text{EH}} = \beta\left[\frac{\pi g^2 r^4}{4\Xi_a\Xi_b} + \frac{\pi g^2 r^2\left(a^2+b^2\right)}{4\Xi_a\Xi_b} - \frac{\pi g^2 r_+^2\left(a^2+b^2+r_+^2\right)}{4\Xi_a\Xi_b}\right]\Bigg|_{r\to+\infty}. \tag{4.57}$$

The Gibbons-Hawking action is found to be

$$\begin{aligned} I_{\text{GH}} = \beta\Bigg[ &-\frac{\pi g^2 r^4}{\Xi_a\Xi_b} + \frac{15\pi r^2\left(\Xi_a+\Xi_b+3/4\right)}{24\Xi_a\Xi_b} && (4.58)\\ &+\frac{\pi\left(a^4g^2-8a^2b^2g^2-9a^2+b^4g^2-9b^2+24m\right)}{24\Xi_a\Xi_b}\Bigg]\Bigg|_{r\to+\infty}. && (4.59) \end{aligned}$$

We can see that the non-regularized action is divergent and given by

$$\begin{aligned} I_{\text{non-reg}}/\beta =& (I_{\text{EH}}+I_{\text{GH}})/\beta \\ =& -\frac{3\pi g^2 r^4}{4\Xi_a\Xi_b} + r^2\left[\frac{\pi g^2\left(a^2+b^2\right)}{4\Xi_a\Xi_b} + \frac{\pi\left(-15a^2g^2-15b^2g^2-18\right)}{24\Xi_a\Xi_b}\right] \\ &+ \text{finite terms}. \end{aligned} \tag{4.60}$$

We now calculate the counterterms action and verify that they cancel these divergences. Evaluation of the counterterms in (3.46) yields

$$I_{\rm ct} = \beta \left[ \frac{3\pi r^2 \left(a^2g^2 + b^2g^2 + 2\right)}{8\Xi_a\Xi_b} + \frac{3\pi g^2 r^4}{4\Xi_a\Xi_b} \right] + I_{\rm ct0}, \tag{4.61}$$

where $I_{\rm ct0}$ is a complicated finite term. The final action using the counterterms method is found by adding this term to the Einstein-Hilbert and Gibbons-Hawking terms in (4.60) and the electromagnetic action $I_{\rm EM}$,

$$\begin{aligned} I = &\frac{\pi\beta}{96g^2\Xi_a\Xi_b}\big[a^4g^4 + b^4g^4 - 24g^4r_+^4 + 24g^2m + 9 \\ &- 3b^2g^2(3+8g^2r_+^2) - a^2g^2\left(17b^2g^2 + 24g^2r_+^2 + 9\right)\big] + I_{\rm EM}. \end{aligned} \tag{4.62}$$

We can simplify the numerator further. Let $\mathcal{N}_{\rm grav}$ be the numerator of (4.62) without the factor $\pi\beta$ and $I_{\rm EM}$. With some re-arrangement of the terms, we re-write this as

$$\mathcal{N}_{\rm grav} = 9\Xi_a 9\Xi_b - 26a^2b^2g^4 + a^4g^4 + b^4g^4 + 24g^2m - 24g^4r_+^2(r_+^2 + b^2 + a^2). \tag{4.63}$$

Now, using

$$(\Xi_a - \Xi_b)^2 = a^4g^4 - 2a^2b^2g^4 + b^4g^4, \tag{4.64}$$

we get

$$\begin{aligned} \mathcal{N}_{\rm grav} &= 9\Xi_a 9\Xi_b + (\Xi_a - \Xi_b)^2 - 24a^2b^2g^4 + 24g^2m - 24g^4r_+^2(r_+^2 + b^2 + a^2) \\ &= 9\Xi_a 9\Xi_b + (\Xi_a - \Xi_b)^2 + 24g^2\left[m - a^2b^2g^2 - g^2r_+^2(r_+^2 + b^2 + a^2)\right] \\ &= 9\Xi_a 9\Xi_b + (\Xi_a - \Xi_b)^2 + 24g^2\left[m - g^2(r_+^2 + a^2)(r_+^2 + b^2)\right]. \end{aligned} \tag{4.65}$$

The final gravitational action is then given by

$$\begin{aligned} I_{\rm grav} &= \frac{\pi\beta\left[9\Xi_a 9\Xi_b + (\Xi_a - \Xi_b)^2\right]}{96g^2\Xi_a\Xi_b} + \frac{\pi\beta\left[m - g^2(r_+^2 + a^2)(r_+^2 + b^2)\right]}{4\Xi_a\Xi_b} \\ &= \beta M_{\rm Casimir} + \frac{\pi\beta\left[m - g^2(r_+^2 + a^2)(r_+^2 + b^2)\right]}{4\Xi_a\Xi_b}. \end{aligned} \tag{4.66}$$

Our result takes the exact same form as the one calculated by Skenderis and Papadimitriou [7] for a non-charged rotating black hole. The difference here of course is that the charge does make an appearance in the $\beta$ term. Nevertheless, it is interesting that the gravitational action maintains the same expression in terms of $\beta$. This also automatically shows that our result does reduce to that in [7] since, in the absence of an electric charge, the temperature takes the same form as that in [7] and the Maxwell action vanishes.

Having shown that the counterterms subtraction method leads to a finite action, we can now calculate the mass and angular momenta using the counterterms method. The lapse function $N$ and foliation metric $\sigma_{ab}$ are defined in §B.3. The latter can be written as

$$\begin{aligned}\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b = & \frac{\rho^2\mathrm{d}^2\theta}{\Delta_\theta} \\ & + \frac{2\sin^2\theta}{\rho^4\Xi_a^2}\Big(a\rho^2\sin^2\theta\left(a^2bg^2q+bq\Xi_a+am\right)-\frac{1}{2}a^2q^2\sin^2\theta \\ & + \frac{1}{2}\rho^4\Xi_a\left(a^2+r^2\right)\Big)\mathrm{d}^2\phi \\ & + \frac{2\cos^2\theta}{\rho^4\Xi_b^2}\Big(b\rho^2\cos^2(\theta)\left(ab^2g^2q+aq\Xi_b+bm\right)-\frac{1}{2}b^2q^2\cos^2\theta \\ & + \frac{1}{2}\rho^4\Xi_b\left(b^2+r^2\right)\Big)\mathrm{d}^2\psi \\ & + \frac{4\sin^2\theta\cos^2\theta}{\rho^4\Xi_a\Xi_b}\Big(\rho^2\left(a^2q\left(b^2g^2+\Xi_b/2\right)+b^2q\Xi_a/2+abm\right) \\ & - \frac{1}{2}abq^2\Big)\mathrm{d}\phi\mathrm{d}\psi,\end{aligned} \tag{4.67}$$

leading to the determinant

$$\begin{aligned}\sigma = & \frac{1}{\rho^2\Xi_a^2\Xi_b^2\Delta_\theta}\Bigg(\sin^2\theta\cos^2\theta\Big(\Xi_a\big(b^2\left(a^2+r^2\right)\cos^2\theta\left(2abg^2\rho^2q+2m\rho^2-q^2\right) \\ & + \Xi_b\big(-2a^2b^2q^2\cos^4\theta+2abq\cos^2\theta\left(a^2\rho^2+abq-b^2\rho^2\right)+\rho^2\left(b^2+r^2\right) \\ & \left(a^2\rho^2+2abq+\rho^2r^2\right)\big)\big)-\frac{1}{2}a^2\Xi_b\sin^2\theta\big(2a^2q^2\Xi_b\cos^2\theta-2\left(b^2+r^2\right) \\ & \left(2abg^2\rho^2q+2m\rho^2-q^2\right)\big)\big)-b^4q^2\Xi_a^2\sin^2\theta\cos^2\theta\Big)\Bigg).\end{aligned} \tag{4.68}$$

The mass is calculated using equation (1.130). To simplify the discussion we will write the mass as the sum of four terms:

$$M = M_{nr1}+M_{nr2}+M_{ct1}+M_{ct2}. \tag{4.69}$$

The first two terms are divergent; they result from the original non-regularized components of the Brown-York quasi-local stress tensor. Evidently, the last two terms are those arising from the counterterms action. The first of those four terms is given by

$$\begin{aligned}M_{nr1} =&\frac{\pi m\left(-3g^2\left(a^2+b^2\right)+a^2g^2\Xi_b+b^2g^2\Xi_a-2\Xi_a\Xi_b+6\right)}{8G\Xi_a^2\Xi_b^2}\\&+\frac{\pi q\left(abg^2\left(-3g^2\left(a^2+b^2\right)+a^2g^2\Xi_b+6\right)+ab^3g^4\Xi_a+2abg^2\Xi_a\Xi_b\right)}{8G\Xi_a^2\Xi_b^2}\\&+\frac{\pi\left(2a^2b^2g^2\Xi_a\Xi_b+2g^2r^2\left(a^2+b^2\right)\Xi_a\Xi_b+2g^2r^4\Xi_a\Xi_b\right)}{8G\Xi_a^2\Xi_b^2}.\end{aligned} \tag{4.70}$$

The second is given by

$$\begin{aligned}M_{nr2} =&\frac{-\pi}{24G\Xi_a\Xi_b}\Big[a^4-g^2+3r^2\big(5g^2\left(a^2+b^2\right)+6\big)+8a^2b^2g^2+9a^2-b^4g^2+9b^2\\&+24g^2r^4-24m\Big].\end{aligned} \tag{4.71}$$

The first part of the counterterms contribution is

$$\begin{aligned}M_{ct1} =&\frac{\pi}{32g^2G\Xi_a^2\Xi_b^2}\Big[12g^2\left(a^2g^2\Xi_b+a^2g^2+b^2g^2-2\right)\left(abg^2q+m\right)+\Xi_a\Big(\Xi_b\big(-a^4g^4+a^2g^2\\&\left(11b^2g^2+18g^2r^2+9\right)+24abg^4q-b^4g^4+9b^2\left(2g^4r^2+g^2\right)+3\big(8g^4r^4+4g^2r^2\\&-1\big)\big)+12b^2g^4\left(abg^2q+m\right)\Big)\Big].\end{aligned} \tag{4.72}$$

And lastly, the second contribution from the counterterms gives

$$M_{ct2} = -\frac{3\pi\left[a^2g^4\left(b^2+r^2\right)+g^2r^2\left(b^2g^2-2\right)-1\right]}{16g^2G\Xi_a\Xi_b}. \tag{4.73}$$

The addition of all these terms gives the mass via the counterterms subtraction method,

$$
\begin{aligned}
M = &-\frac{\pi\left(-a^4g^2+3r^2\left(5g^2\left(a^2+b^2\right)+6\right)+8a^2b^2g^2+9a^2-b^4g^2\right)}{24\Xi_a\Xi_b}\\
&-\frac{\pi\left(9b^2+24g^2r^4-24m\right)}{24\Xi_a\Xi_b}\\
&+\frac{\pi}{8\Xi_a^2\Xi_b^2}\Bigg(\Big(-3g^2\left(a^2+b^2\right)+a^2g^2\Xi_b+6\Big)\left(abg^2q+m\right)\\
&+\Xi_a\Big(2\Xi_b\left(g^2\left(r^2\left(a^2+b^2\right)+ab(ab+q)+r^4\right)-m\right)\\
&+b^2g^2\left(abg^2q+m\right)\Big)\Bigg)\\
&+\frac{\pi}{32g^2\Xi_a^2\Xi_b^2}\Big(12g^2\left(g^2\left(a^2+b^2\right)+a^2g^2\Xi_b-2\right)\left(abg^2q+m\right)\\
&+\Xi_a\Big(\Xi_b\Big(g^2\big(a^4\left(-g^2\right)+6r^2\left(3g^2\left(a^2+b^2\right)+2\right)\\
&+11a^2b^2g^2+9a^2+24abg^2q-b^4g^2+9b^2+24g^2r^4\big)-3\Big)\\
&+12b^2g^4\left(abg^2q+m\right)\Big)\Big).
\end{aligned} \tag{4.74}
$$

This expression, which is the first that we got by adding the different contributions to the mass, is evidently very messy. We can use the definitions of $\Xi_a$ and $\Xi_b$ to simplify it further, first arriving at

$$
\begin{aligned}
M = &\frac{\pi}{96g^2\Xi_a^2\Xi_b^2}\Big(48a^2g^4\Xi_b\left(abg^2q+m\right)+\Xi_a\Big(\Xi_b\big(a^4g^4+a^2g^2\left(7b^2g^2-9\right)\\
&+96abg^4q+b^4g^4-9b^2g^2+72g^2m+9\big)+48b^2g^4\left(abg^2q+m\right)\Big)\Big).
\end{aligned} \tag{4.75}
$$

Expression (4.75) is already a huge improvement over (4.74) and we are encouraged to keep going with our simplification efforts. We will begin by dividing the mass into three terms: $M_m$, which contains the polynomial in $m$, $M_q$, which contains the polynomial in $q$, and $M_c$ which contains the rest of the terms. These quantities are given by

$$
\begin{aligned}
M_m &= \pi m\left(48a^2g^4\Xi_b+48b^2g^4\Xi_a+72g^2\Xi_a\Xi_b\right)/96g^2\Xi_a^2\Xi_b^2,\\
M_q &= \pi q\left(48a^3bg^6\Xi_b+48ab^3g^6\Xi_a+96abg^4\Xi_a\Xi_b\right)/96g^2\Xi_a^2\Xi_b^2,\\
M_c &= \big(9-9a^2g^2-18b^2g^2+a^4g^4+16a^2b^2g^4+10b^4g^4\\
&\quad -a^4b^2g^6-7a^2b^4g^6-b^6g^6\big)/96g^2\Xi_a^2\Xi_b^2.
\end{aligned} \tag{4.76}
$$

The first two terms combine to give

$$
\begin{aligned}
M_m + M_q &= \frac{\pi m \left(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b\right) + 2\pi a b g^2 q \left(\Xi_a + \Xi_b\right)}{4\Xi_a^2 \Xi_b^2} \\
&= M_0.
\end{aligned}
\tag{4.77}
$$

Amazingly, this quantity corresponds to the black hole mass calculated in [10] by integrating the first law.

Let $\mathcal{N}$ be the numerator of $M_c$. Let us see if we can simplify it further. First we expand the expression to

$$
\mathcal{N} = \pi \Xi_a \Xi_b \big(9 - 9b^2 g^2 + a^4 g^4 + b^4 g^4 - 9a^2 g^2 + 7a^2 b^2 g^4\big) \tag{4.78}
$$

We can use the fact that

$$
a^4 g^4 + b^4 g^4 = \left(\Xi_a - \Xi_b\right)^2 + 2a^2 b^2 g^4, \tag{4.79}
$$

to re-write $\mathcal{N}$ as

$$
\begin{aligned}
\mathcal{N} &= \pi \Xi_a \Xi_b \big[9\left(1 - a^2 g^2 - b^2 g^2 + a^2 b^2 g^4\right) + \left(\Xi_a - \Xi_b\right)^2\big] \\
&= \pi \Xi_a \Xi_b \big[9\Xi_a \Xi_b + \left(\Xi_a - \Xi_b\right)^2\big].
\end{aligned}
\tag{4.80}
$$

Finally, the term $M_c$ can be written as

$$
\boxed{M_c = \frac{\pi\left[9\Xi_a \Xi_b + \left(\Xi_a - \Xi_b\right)^2\right]}{96 G g^2 \Xi_a \Xi_b}}, \tag{4.81}
$$

where we have reinstated the gravitational constant $G$ (originally a factor in the denominator of (3.53)). The full expression for the mass calculated via the counterterms method can now we written as

$$
\boxed{M = \frac{\pi m \left(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b\right) + 2\pi a b g^2 q \left(\Xi_a + \Xi_b\right)}{4 G \Xi_a^2 \Xi_b^2} + \frac{\pi\left[9\Xi_a \Xi_b + \left(\Xi_a - \Xi_b\right)^2\right]}{96 G g^2 \Xi_a \Xi_b}}. \tag{4.82}
$$

Note that the quantity $M_c$ is the background energy of the spacetime: it is the value which the total energy (4.82) reduces to in the absence of the black hole when $m = 0$, $q = 0$. When the two rotation parameters are set to 0, this background energy reduces to the background energy of the pure non-rotating AdS$_5$ spacetime given in [38] by

$$M_{\text{BY}}\Big|_{a=b=0} = \frac{3\pi}{32g^2G}. \tag{4.83}$$

Note also that the expression of the vacuum energy (4.81) does not depend on $q$. This of course makes sense since the vacuum energy should be oblivious to the black hole charge. This also means that we should expect our expression for the background energy to match that of a rotating, non-charged black hole in AdS$_5$. The latter is given in reference [7] and is in complete agreement with our finding.

### 4.3.2 Conformal Anomaly Calculations

As predicted by the AdS/CFT correspondence, the conformal anomaly calculated from the conformal field theory on the boundary should match the analogous calculation in the gravitational theory. In this section we will aim the verify this prediction.

The CFT metric is found by taking our boundary metric and removing a divergent conformal factor $g^2r^2$ [9]. The boundary metric is given by

$$ds^2_{\text{Boundary}} = g^2r^2\left[\frac{\Delta_\theta(\theta)}{\Xi_a\Xi_b}\mathrm{d}t^2 + \frac{1}{g^2\Delta_\theta(\theta)}\mathrm{d}\theta^2 + \frac{\sin^2\theta}{g^2\Xi_a}\mathrm{d}\phi^2 + \frac{\cos^2\theta}{g^2\Xi_b}\mathrm{d}\psi^2\right]. \tag{4.84}$$

The CFT metric can therefore be written as

$$ds^2_{\text{CFT}} = \frac{\Delta_\theta(\theta)}{\Xi_a\Xi_b}\mathrm{d}t^2 + \frac{1}{g^2\Delta_\theta(\theta)}\mathrm{d}\theta^2 + \frac{\sin^2\theta}{g^2\Xi_a}\mathrm{d}\phi^2 + \frac{\cos^2\theta}{g^2\Xi_b}\mathrm{d}\psi^2. \tag{4.85}$$

We are mainly interested in calculating the expectation value of the renormalized CFT stress tensor, and from it the conformal anomaly and Casimir energy of that theory. Since this thesis is related to General Relativity and not Conformal Field Theory, we will not go into the details of how the expectation value of stress tensors are defined and renormalized in quantum field theories. These details are outlined in Chapter 6 of reference [51]. Here we will directly use the formula given in [51] for the renormalized stress tensor:

$$\langle\hat{T}_{ab}\rangle = -\sum_s \beta^s H^{(3)}_{ab}, \tag{4.86}$$

where the summation is over the possible fields of the theory, with $s = 0, \frac{1}{2}, 1$ standing for scalar, fermion and gauge fields respectively. The values of the coefficients $\beta^s$ are given in [52] by

$$\beta^0 = -\frac{1}{2880\pi^2} N^0, \tag{4.87}$$

$$\beta^{\frac{1}{2}} = -\frac{1}{2880\pi^2} N^{\frac{1}{2}}, \tag{4.88}$$

$$\beta^1 = -\frac{1}{2880\pi^2} N^1, \tag{4.89}$$

where $N^i$ is the number of fields of spin $i$. It turns out that these numbers are [9] $N^0 = 6N^2$, $N^{\frac{1}{2}} = 4N^2$ and $N^1 = N^2$. The tensor $H^{(3)}_{ab}$ is given in [51] by

$$H^{(3)}_{ab} = \frac{1}{12}\mathcal{R}^2\gamma_{ab} - \mathcal{R}^{cd}\mathcal{R}_{cadb}. \tag{4.90}$$

As before, $\gamma_{ab}$ is the CFT metric tensor and $\mathcal{R}_{abcd}$, $\mathcal{R}_{ab}$ and $\mathcal{R}$ are the Riemann tensor, Ricci tensor and Ricci scalar of the CFT. Evaluation of this stress tensor yields the conformal anomaly

$$\begin{aligned}
\langle \hat{T}^a{}_a \rangle &= -\frac{3N^2(a-b)g^4(a+b)}{8\pi^2}\Big[2g^4(a-b)(a+b)\cos(\theta)^4 + \Big(\big(\frac{8b^2}{3} - \frac{4a^2}{3}\big)g^4 - \frac{4g^2}{3}\Big)\cos(\theta)^2 \\
&\quad - \frac{2g^2(bg-1)(bg+1)}{3}\Big] + \mathcal{O}\left(r^{-2}\right) \\
&= -\frac{N^2\left(a^2-b^2\right)\left(3\cos^4\theta a^2 - 3\cos^4\theta b^2 - 2a^2\cos^2\theta + 4b^2\cos^2\theta - \frac{2\cos^2\theta}{g^2} - b^2\right)g^8}{4\pi^2} \\
&\quad + \frac{g^6}{4\pi^2} \\
&= -\frac{\left(a^2-b^2\right)N^2g^6}{4\pi^2}\Big[a^2g^2\cos^2\theta\left(3\cos^2\theta - 2\right) + b^2g^2\left(\cos^2\theta\left(-3\cos^2\theta + 4\right) - 1\right) \\
&\quad - \cos 2\theta\Big] \qquad (4.91) \\
&= -\frac{\left(a^2-b^2\right)N^2g^6\Big[3g^2\left(a^2-b^2\right)\cos^4\theta - 2\cos^2\theta\left(a^2g^2 - 2b^2g^2 + 1\right) - b^2g^2 + 1\Big]}{4\pi^2}.
\end{aligned} \tag{4.92}$$

Our result (particularly the penultimate expression) matches that found in [45] for a neutral rotating black hole in AdS$_5$.

The gravitational quasi-local stress tensor is evidently related to the CFT stress tensor by ([9])

$$\sqrt{-\gamma}\,\gamma_{ab}\langle\hat{T}^{bc}\rangle = \lim_{r\to\infty}\sqrt{-h}\,h_{ab}T^{bc}. \tag{4.93}$$

We therefore expect the trace of the gravitational tensor to be related to that of the CFT stress tensor by a factor $\lim_{r\to\infty}\sqrt{h/\gamma}$. This factor is found to be

$$\lim_{r\to\infty}\sqrt{h/\gamma} = g^4 r^4. \tag{4.94}$$

So terms up to order $\mathcal{O}(r^{-4})$ will survive when multiplied by this factor. Evaluation of the gravitational quasi-local tensor gives

$$\begin{aligned} T^a{}_a &= \frac{1}{8\pi g G r^4 \Xi_a}\Big[ -\Xi_b\left(a^4 g^2 + b^2\right) + a^2\left(1 - b^4 g^4\right) + 3g^2\left(a^2 - b^2\right)^2 \Xi_a \cos^4\theta + 2\left(a^2 - b^2\right) \\ &\qquad \left(a^2 g^2 - 1\right)\cos^2\theta\big(a^2 g^2 - 2b^2 g^2 + 1\big)\Big] + \mathcal{O}(r^{-6}) \\ &= -\frac{\left(a^2 - b^2\right)\Big[3g^2\left(a^2 - b^2\right)\cos^4\theta - 2\cos^2\theta\left(a^2 g^2 - 2b^2 g^2 + 1\right) - b^2 g^2 + 1\Big]}{8\pi g G r^4}. \end{aligned}$$

Multiplying by the conformal factor $g^4 r^4$,

$$\boxed{g^4 r^4 T^a{}_a = -\frac{g^3\left(a^2 - b^2\right)\Big[3g^2\left(a^2 - b^2\right)\cos^4\theta - 2\cos^2\theta\left(a^2 g^2 - 2b^2 g^2 + 1\right) - b^2 g^2 + 1\Big]}{8\pi G}}. \tag{4.95}$$

We have mentioned that in the framework of the AdS/CFT Correspondence, $N^2$ is the quantity dual to the gravity-side quantity $\pi/2g^3 G$. Using this we can easily see that this yields the same expression as in (4.91). We have therefore shown that the conformal anomaly calculated from the gravity theory is exactly equivalent to that calculated in the dual conformal field theory defined on the boundary of the spacetime.

### 4.3.3 Casimir Energy Calculations

We have shown that the background energy of the gravitational theory is given by $M_c$. We would now like to see if it does in fact correspond to the vacuum energy of the dual conformal field theory on the boundary. The vacuum energy is found using the formula ([9])

$$E_{\text{Casimir}} = \sum_{s=0,\frac{1}{2},1} N^s \int_{S^3} \mathrm{d}^3x\sqrt{\sigma}\,\chi^a\langle\hat{T}^s_{ab}\rangle u^b. \tag{4.96}$$

Here the summation is again over the possible fields of the theory. $\chi^a$ and $u^a$ are still the timelike Killing vector and unit normal vector as before. $\sigma$ is also the conformal foliation metric of the boundary. The conformal foliation metric is now found using

$$\sigma_{ab} = g^2r^2\left(g_{ab} + u_a u_b\right), \tag{4.97}$$

leading to the line element

$$\sigma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\left(a^2\cos^2\theta + b^2\sin^2\theta + r^2\right)\mathrm{d}\theta^2}{g^2r^2\Delta_\theta} + \frac{\sin^2\theta\mathrm{d}\phi^2}{g^2\Xi_a} + \frac{\cos^2\theta\mathrm{d}\psi^2}{g^2\Xi_b}, \tag{4.98}$$

and the determinant

$$\sigma = \frac{\left(a^2\cos^2\theta + b^2\sin^2\theta + r^2\right)\sin^2\theta\cos^2\theta}{g^6r^2\Delta_\theta\Xi_a\Xi_b}. \tag{4.99}$$

Direct evaluation of the integral in (4.96) yields

$$E_{\text{Casimir}} = \frac{\left(g^4a^4 + 7g^4b^2a^2 + b^4g^4 - 9a^2g^2 - 9b^2g^2 + 9\right)N^2\sqrt{\frac{1}{\Xi_b}g^2r}}{48\Xi_a\sqrt{\Xi_b}}. \tag{4.100}$$

We can simplify this by hand to arrive at

$$E_{\text{Casimir}} = \frac{\left(g^4a^4 + 7g^4b^2a^2 + b^4g^4 - 9a^2g^2 - 9b^2g^2 + 9\right)N^2 g}{48\Xi_a\Xi_b}. \tag{4.101}$$

Then, combining the second, fourth and fifth terms, we can re-write this as

$$E_{\text{Casimir}} = \frac{N^2g\left(a^4g^4 + b^4g^4 - 2a^2b^2g^4 + 9\Xi_a\Xi_b\right)}{48\Xi_a\Xi_b}.$$

Using equation (4.64), we get a final expression for the Casimir energy given by

$$\boxed{E_{\text{Casimir}} = \frac{N^2 g\left[9\Xi_a\Xi_b + (\Xi_a - \Xi_b)^2\right]}{48\Xi_a\Xi_b}}. \tag{4.102}$$

Making the identification $\pi/(2Gg^3) \leftrightarrow N^2$, it is easy to see that the Casimir energy in (4.102) is identical to that in (4.81). We conclude that the background energy of the gravitational theory is identical to the vacuum energy of the dual conformal field theory residing on its boundary.

### 4.3.4 Angular Momenta Calculation and First Law Verification

The final quantities to calculate are the angular momenta. Calculation of the angular momentum using the Komar integral in this case yields no divergences [10]. Nevertheless, for consistency, we will calculate it using the counterterms method to check if this gives the same value as the Komar integral. The angular momentum is given by the Brown-York formalism in (1.131), where to get $J_a$ we set $\eta = \eta_a$ and to get $J_b$ we set $\eta = \eta_b$. Like we did for the mass, we divide $J_a$ into four parts associated with each of the four parts of the full stress tensor (3.53),

$$J_a = J_{nr1} + J_{nr2} + J_{ct1} + J_{ct2}. \tag{4.103}$$

The last three terms yield 0. Since we expect the angular momentum to be finite, it is not surprising that the counterterms' contribution vanishes, since there is no divergence to regularize. Before integration, the first part gives an integrand

$$j_a = \frac{\sin^3\theta\cos\theta\left[am + b\left(a^2g^2q + \frac{1}{2}q\Xi_a\right)\right]}{2\Xi_a^2\Xi_b}. \tag{4.104}$$

Calculating the integration and simplifying the expression by hand, we get

$$\begin{aligned}
J_a &= \int_{S^3} \frac{\sin^3\theta\cos\theta\left[am + b\left(a^2g^2q + \frac{1}{2}q\Xi_a\right)\right]}{2\pi\Xi_a^2\Xi_b}\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi \\
&= \frac{\pi^2\left(a^2bg^2q + 2am + bq\right)}{4\pi\Xi_a^2\Xi_b^2} \\
&= \frac{\pi\left[2am + qb\left(1 + a^2g^2\right)\right]}{4\Xi_a^2\Xi_b}.
\end{aligned}$$

This is equal to the angular momentum calculated in [10] using the Komar integral. We find a similar result for $J_b$,

$$
\begin{aligned}
J_b &= \int_{S^3} \frac{\sin^3\theta\cos\theta\left[bm + a\left(b^2g^2q + \frac{1}{2}q\Xi_b\right)\right]}{2\pi\Xi_a\Xi_b^2}\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi \\
&= \frac{\pi^2\left(b^2ag^2q + 2bm + aq\right)}{4\pi\Xi_a\Xi_b^2} \\
&= \frac{\pi\left[2bm + qa\left(1 + b^2g^2\right)\right]}{4\Xi_a\Xi_b^2}.
\end{aligned}
$$

This is also equal to the expression given in [10] by (4.18).

We will now redirect our attention back to the first law. Since we have shown that the term $M_c$ is in fact the Casimir energy, we have all the terms in the modified first law (4.54) presented by Skenderis and Papadimitriou.

Normally we should verify four equations of the form (4.25). However we can be smart about our approach and save some effort. Note that $M_c$ does not depend on $r_+$ or $q$. Hence it will not contribute is any variations with respect to these parameters. Now, since our expression for the mass in (4.82) is identical to that of $M_0$ (for which we have verified the first law) plus $M_c$, the variations with respect to $r_+$ and $q$ will yield the same result as those for $M_0$.

For the variations with respect to $a$ and $b$ the Casimir energy does contribute on the left-hand side. However, here the right-hand side has a corresponding term $\delta M_{\text{Casimir}}$, so this effect is cancelled. We are hence left with the variation of $M_0$ on the right-hand side and the old right-hand side of the first $T\mathrm{d}S + \Omega_a\mathrm{d}J_a + \Omega_b\mathrm{d}J_b + \Phi\mathrm{d}Q$. Therefore, the first law proposed by Skenderis and Papadimitriou does hold for the general charged rotating $\mathrm{AdS}_5$ black hole presented in [10].

# Conclusion

We have studied the thermodynamics of the solution for charged black holes with Chern-Simons term and two independent rotation parameters presented in [10]. We have computed the electric potential for this solution and shown that the thermodynamical quantities do satisfy the traditional form of the first law of black hole thermodynamics as well as the relations between the Gibbs potential and the extensive quantities of the first law. Nevertheless, the mass presented in [10] does present several disadvantages. Since the counterterms subtraction method was not used, we cannot retrieve the background energy of the solution to compare it to the dual field theory residing on the AdS boundary. We also cannot calculate the conformal anomaly of the conformal field theory using the gravitational theory's energy momentum tensor, since the latter was not regularized. Furthermore, in calculating the mass by integrating the first law, we cannot use it to check if the first law does in fact hold for this solution. Finally, we noted that this method prevents us from relating our energy to a time-translational Killing vector.

We have studied the thermodynamics of the solution for charged black holes with Chern-Simons term and two independent rotation parameters presented in [10]. We have computed the electric potential for this solution and shown that the thermodynamical quantities do satisfy the traditional form of the first law of black hole thermodynamics as well as the relations between the Gibbs potential and the extensive quantities of the first law. Nevertheless, the mass presented in [10] does present several disadvantages. First, in calculating the mass by integrating the first law, we cannot use the resulting expression for the mass to check if the first law does in fact hold for this solution. Another issue with the mass expression in [10] that we have discussed is that this method prevents us from relating our energy to a time-translational Killing vector. In addition, since the counterterms subtraction technique was not used, we cannot retrieve the background energy of the solution to compare it to the dual field theory residing on the AdS boundary. We also cannot calculate the conformal anomaly of the conformal field theory using the gravitational theory's energy momentum tensor, since the latter was not regularized.

Inspired by the results of Skenderis and Papadimitriou in [7], we have proceeded with the counterterms subtraction method and found an expression for the mass which does satisfy the modified first law of thermodynamics presented in [7]. The general form of the first law in [7] was derived for an action without a Chern-Simons term, and the procedure was applied to a neutral black hole. Our findings show that the general form of the first law obtained in [7] still holds for a charged black hole with Maxwell and Chern-Simons actions.

The expression found for the energy using the counterterms method leads to a finite value for the background energy of the AdS spacetime. Unsurprisingly, it was found that this energy does not depend on the electric charge of the black hole. In fact, it is exactly equal to the background energy of the neutral rotating solution [7]. With this expression in hand, we have calculated the vacuum energy of the conformal field theory on the spacetime boundary and verified that it is exactly equal to the background energy of the gravitational theory, as predicted by the AdS/CFT correspondence. We have also shown that calculation of the conformal anomaly from the gravitational side exactly matches that obtained by performing the calculation in the conformal field theory. The gravity calculations were done using the finite quasi-local stress tensor obtained from the action regularized by the counterterms subtraction method.

Interestingly, we have also found that the gravitational part of the action regularized using the counterterms subtraction method takes the exact form as the complete action of the non-charged solution [7], with the charge only appearing in the explicit expression of the temperature. We have also shown that the Brown-York formula for the angular momenta does yield finite quantities that do not need regularization and that, accordingly, the counterterms do vanish. The angular momenta calculated using the Brown-York method were found to be exactly equal to those calculated using the Komar integral in [10] for this solution.

# Appendix A

# Mathematical Operations

**Wedge Product:** The wedge product of a $p$-form $A$ and a $q$-form $B$ is a $(p+q)$-form given by ([11])

$$(A \wedge B)_{a_1 a_2 \ldots a_{p+q}} := \frac{(p+q)!}{p!q!} A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}. \tag{A.1}$$

From this definition it is easy to see that the wedge product is associative:

$$(A \wedge B) \wedge C = A \wedge (B \wedge C). \tag{A.2}$$

**Exterior Derivative:** The exterior derivative of a $p$-form $A$ is a $(p+1)$-form $\mathrm{d}A$. This is given in a specific coordinate basis by ([15])

$$(\mathrm{d}A)_{\mu_1 \mu_2 \ldots \mu_{p+1}} := (p+1) \nabla_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}. \tag{A.3}$$

A differential form $A$ is said to be **closed** if $\mathrm{d}A = 0$. A $p$-form $B$ is said to be **exact** if there exists a $(p-1)$-form $C \,|\, B = \mathrm{d}C$.

**Hodge Star Operator:** Given a $p$-form $A$ on a $D-$dimensional manifold, we define the Hodge dual as the $(D-p)$-form given by

$$(\star A)_{a_1 \ldots a_{D-p}} := \frac{1}{p!} \epsilon^{b_1 \ldots b_p}{}_{a_1 \ldots a_{D-p}} A_{b_1 \ldots b_p}, \tag{A.4}$$

where $\epsilon$ is the Levi-Civita tensor given by

$$\epsilon^{a_1 \ldots a_n} = \frac{1}{\sqrt{|g|}} \tilde{\epsilon}^{a_1 \ldots a_n}. \tag{A.5}$$

Here $\tilde{\epsilon}^{a_1 \ldots a_n}$ stands for the Levi-Civita alternating symbol, and $g$ is the determinant of the metric. It follows that

$$\star \star A = (-1)^{p(D-p)+1} A \tag{A.6}$$

**Proof:** Direct application of a second Hodge star operator yields

$$\star(\star A) = \frac{1}{p!(D-p)!} \epsilon^{a_1 .. a_{D-p}}{}_{c_1 \ldots c_p} \epsilon^{b_1 \ldots b_p}{}_{a_1 \ldots a_{D-p}} A_{b_1 \ldots b_p}. \tag{A.7}$$

We can then use a useful formula relating the contraction of Levi-Civita tensors [11]:

$$\epsilon^{a_1 .. a_{D-p} c_1 \ldots c_p} \epsilon_{b_1 \ldots b_p a_1 \ldots a_{D-p}} = -p!(D-p)! \delta^{c_1 \ldots c_p}_{[b_1 \ldots b_p]}. \tag{A.8}$$

To convert (A.7) to something like (A.8) we can raise all the $c_n$ indices as upper $d_n$ indices and multiply by $g_{c_1 d_1} \ldots . g_{c_p d_p}$. Likewise we can lower all the $b_n$ indices in the second Levi-Civita tensor.

$$\star \star A = \frac{1}{p!(D-p)!} \epsilon^{a_1 .. a_{D-p} d_1 \ldots d_p} \epsilon_{e_1 \ldots e_p a_1 \ldots a_{D-p}} \, g_{c_1 d_1} \ldots g_{c_p d_p} g^{b_1 e_1} \ldots . g^{b_p e_p} A_{b_1 \ldots b_p}. \tag{A.9}$$

We now would like to swap all the $c_n$ and $a_n$ indices in the second Levi-Civita tensor. Evidently this gives a factor of $(-1)^{p(D-p)}$,

$$
\begin{aligned}
\star\star A &= \frac{(-1)^{p(D-p)+1}}{p!(D-p)!}\epsilon^{a_1..a_{D-p}d_1...d_p}\epsilon_{a_1...a_{D-p}e_1...e_p}A_{b_1...b_p}\,g_{c_1d_1}...g_{c_pd_p} \\
&\quad g^{b_1e_1}....g^{b_pe_p} \\
&= \frac{(-1)^{p(D-p)+1}p!(D-p)!}{p!(D-p)!}\delta^{c_1...c_p}_{[b_1...b_p]}g_{c_1d_1}...g_{c_pd_p}\,g^{b_1e_1}....g^{b_pe_p}A_{b_1...b_p} \\
&= (-1)^{p(D-p)+1}\delta^{[e_1...e_p]}_{d_1...d_p}A_{b_1...b_p} \\
&= \frac{(-1)^{p(D-p)+1}}{p!}A_{e_1...e_p}(\text{even permutations of } \delta \\
&\qquad\qquad - \text{odd permutations of } \delta).
\end{aligned}
\tag{A.10}
$$

The presence of the term $\delta^{...e_n...}_{...b_n...}$ will simply re-label $A_{...b_n...}$ to $A_{...e_n...}$. Since $A$ is a differential form, all even permutations of any two indices of $A_{e_1...e_p}$ will yield the same tensor, while all odd permutations will yield the same tensor multiplied by $-1$. Since the total number of permutations is $p!$, this simply gives

$$
\star\star A = (-1)^{p(D-p)+1}A_{e_1...e_p}\frac{p!}{p!} \tag{A.11}
$$

$$
= (-1)^{p(D-p)+1}A. \tag{A.12}
$$

□

Another important identity related to the Hodge operator is

$$
\star\mathrm{d}\star A = -(-1)^{p(D-p)}\nabla^b A_{a_1...a_{p-1}b}. \tag{A.13}
$$

**Proof:** Direct application of the exterior derivative yields

$$
\begin{aligned}
\mathrm{d}(\star A) &= \frac{(D-p+1)}{p!}\nabla_{[c}\epsilon^{b_1...b_p}{}_{a_1...a_{D-p}]}A_{b_1...b_p} \\
&= \frac{(D-p+1)}{p!}\epsilon^{b_1...b_p}{}_{[a_1...a_{D-p}}\nabla_{c]}A_{b_1...b_p},
\end{aligned}
\tag{A.14}
$$

where I have used the fact that $\nabla\epsilon = 0$. Applying another Hodge operator,

$$\star(\mathrm{d}\star A) = \frac{(D-p+1)}{(D-p+1)!p!}\epsilon^{a_1\dots a_{D-p}c}{}_{d_1\dots d_{p-1}}\epsilon^{b_1\dots b_p}{}_{[a_1\dots a_{D-p}}\nabla_{c]}A_{b_1\dots b_p}$$
$$= \frac{1}{(D-p)!p!}\epsilon_{a_1\dots a_{D-p}cd_1\dots d_{p-1}}\epsilon^{b_1\dots b_p[a_1\dots a_{D-p}}\nabla^{c]}A_{b_1\dots b_p} \tag{A.15}$$

The tensor $\epsilon^{b_1\dots b_p[a_1\dots a_{D-p}}\nabla^{c]}$ is contracted with a Levi-Civita tensor, therefore only its totally anti-symmetric part will survive and we can remove the anti-symmetrization since it is now redundant.[1]

$$\star\mathrm{d}\star A = \frac{1}{(D-p)!p!}\epsilon_{a_1\dots a_{D-p}cd_1\dots d_{p-1}}\epsilon^{b_1\dots b_p a_1\dots a_{D-p}}\nabla^c A_{b_1\dots b_p}. \tag{A.16}$$

Following the same alternation trick from the previous proof for the indices of the second Levi-Civita tensor,

$$\star\mathrm{d}\star A = \frac{(-1)^{p(D-p)}}{(D-p)!p!}\epsilon_{a_1\dots a_{D-p}cd_1\dots d_{p-1}}\epsilon^{a_1\dots a_{D-p}b_1\dots b_p}\nabla^c A_{b_1\dots b_p}. \tag{A.17}$$

Then again we use equation (A.8) and re-write the previous expression as

$$\star\mathrm{d}\star A = \frac{-(D-p)!p!(-1)^{p(D-p)}}{(D-p)!p!}\delta^{b_1\dots b_p}_{d1\dots d_{p-1}c}\nabla^c A_{b_1\dots b_p}$$
$$= (-1)^{p(D-p)+1}\nabla^c\delta^{b_1\dots b_p}_{[d1\dots d_{p-1}c]}A_{b_1\dots b_p}$$
$$= (-1)^{p(D-p)+1}\nabla^c A_{b_1\dots b_{p-1}c}. \tag{A.18}$$

In the second line I have used the fact that the covariant derivative of the Kronecker delta vanishes, and in the third line I have used the same permutations trick when contracting $\delta^{b_1\dots b_p}_{d1\dots d_{p-1}c}$ with $A_{b_1\dots b_p}$ that I have used in the previous proof.

□

**Lie Derivative:** If a vector $V^a(x)$ is defined on a manifold $\mathcal{M}$, it can be used to relate two infinitesimally close points $P$ and $\bar{P}$ in $\mathcal{M}$. If the coordinates of $P$ and $\bar{P}$ are $x^a$ and $\bar{x}^a$ respectively, then we can write

[1]Special thanks to Ahmed Hemdan for pointing out that this trick would be helpful here.

$$\bar{x}^a = x^a - \varepsilon V^a(x), \tag{A.19}$$

where $\varepsilon$ is an infinitesimally small constant. Note that this is a change of points on the manifold $\mathcal{M}$, not a change of coordinates. Given a vector field $X^a$ in $\mathcal{M}$, it can be expressed using $\bar{x}$ at $\bar{P}$ by

$$\begin{aligned}\bar{X}^a(\bar{x}) &= \frac{\partial \bar{x}^a}{\partial x^b} X^b(x) \\ &= \left(\delta^a{}_b - \varepsilon \partial_b V^a\right) X^b(x) \\ &= X^a(x) - \varepsilon \partial_b V^a X^b(x).\end{aligned} \tag{A.20}$$

In the last two lines all partial derivatives are with respect to $x$, i.e.

$$\partial_c \equiv \frac{\partial}{\partial x^c}. \tag{A.21}$$

The careful reader will notice that the quantity $\bar{X}(\bar{x}) - X(x)$ is not a vector because it is the difference of two vectors *at two different points* [53]. We can expand $\bar{X}\left(\bar{x}\right)$:

$$\begin{aligned}\bar{X}^a\left(\bar{x}\right) &= \bar{X}^a(x) + \left(\bar{x} - x\right)^b \partial_b \bar{X}^a(x) + ... \\ &= \bar{X}^a(x) - \varepsilon V^b(x) \partial_b \bar{X}^a + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{A.22}$$

Combining (A.20) with (A.22), we have

$$X^a(x) - \bar{X}^a(x) = -\varepsilon V^b \partial_b \bar{X}(x) + \varepsilon \partial_b V^a X^b(x) + \mathcal{O}(\varepsilon^2). \tag{A.23}$$

and

$$\lim_{\varepsilon \to 0} \left( \frac{X^a(x) - \bar{X}^a(x)}{\varepsilon} \right) = -V^b \partial_b X^a(x) + \partial_b V^a X^b. \tag{A.24}$$

This quantity is called the **Lie derivative** and is denoted $\mathcal{L}_V X$. It is clear that the Lie derivative $\mathcal{L}_V X$ quantifies the change of the vector field $X$ along a flow defined by the vector field $V$. We can extend this notion to incorporate covariant derivatives:

$$\mathcal{L}_V X^a = -V^b \nabla_b X^a(x) + \nabla V^a X^b. \tag{A.25}$$

We can also generalize it to include tensors of any rank as

$$\mathcal{L}_V T^{a_1..a_p}{}_{b_1...b_q} = \lim_{\varepsilon \to 0} \frac{T^{a_1..a_p}{}_{b_1...b_q}(x) - \bar{T}^{a_1..a_p}{}_{b_1...b_q}(x)}{\varepsilon}. \tag{A.26}$$

In the general case this gives

$$\begin{aligned}(\mathcal{L}_V T)^{a_1...a_p}{}_{b_1...b_q} = & V^c(\partial_c T^{a_1...a_p}{}_{b_1...b_q}) \\ & - (\partial_c V^{a_1})T^{ca_2...a_p}{}_{b_1...b_q} - \ldots - (\partial_c V^{a_p})T^{a_1...a_{p-1}c}{}_{b_1...b_q} \\ & + (\partial_{b_1} V^c)T^{a_1...a_p}{}_{cb_2...b_q} + \ldots + (\partial_{b_q} V^c)T^{a_1...a_p}{}_{b_1...b_{q-1}c}.\end{aligned} \tag{A.27}$$

For example, in the case of a second-rank tensor with two lower indices, this gives

$$\mathcal{L}_V T_{ab} = T_{ab}\partial_c V^c + T_{cb}\partial_a V^c + T_{ac}\partial_b V^c. \tag{A.28}$$

And for a tensor with two upper indices this gives

$$\mathcal{L}_V T^{ab} = T^{ab}\partial_c V^c - T^{cb}\partial_c V^a - T^{ac}\partial_a V^b. \tag{A.29}$$

# Appendix B

# Hypersurfaces, Extrinsic Curvature and Foliation

## B.1 Hypersurfaces

We will start by recalling the concept of *tangent spaces*. Given a manifold $\mathcal{M}$, we can define at each point $p \in \mathcal{M}$ a vector space $T_p(\mathcal{M})$ containing all vectors that pass through $p$ and which are tangent to the manifold $\mathcal{M}$.

The hypersurface is a generalization of the concept of two-dimensional surfaces. A hypersurface $\Sigma$ is a $(D-1)$-dimensions submanifold of an $n$-dimensional manifold $\mathcal{M}$ [54]. Given a function $f : \mathcal{M} \to \mathbb{R}$, a hypersurface $\Sigma$ can be defined by constraining $f(x)$ to a constant value $f_0$. Recall from multivariable calculus that the gradient of a function constraining a surface is perpendicular to that surface. We have here a similar result. Let $\zeta^a = \nabla^a f(x)$. This vector will be perpendicular to $\Sigma$ in the sense that,

$$\forall\, V^a \in T_p(\Sigma) \subset T_p(\mathcal{M}),\; V^a \zeta_a = 0. \tag{B.1}$$

If $\zeta^a$ is timelike (respectively spacelike), then the hypersurface is said to be spacelike (respectively timelike). Otherwise, if $\zeta^a$ is null then the hypersurface is said to be null. Here we will mostly discuss the first two types. We can define a normalized orthogonal vector to $\Sigma$ by

$$n^a = \frac{\zeta^a}{|\zeta^b \zeta_b|^{\frac{1}{2}}}. \tag{B.2}$$

To see how certain mathematical operations can be done on the hypersurface, we define the

**projection tensor** for $\Sigma$ by

$$P_{ab} = g_{ab} - s\, n_a n_b, \tag{B.3}$$

where $s = n^a n_a$. It is easy to see that this metric projects any vector $V^a \in T_p(\mathcal{M})$ onto $\Sigma$. To see this, note that the vector can be written as the sum of two vector $V_\perp^a$ and $V_\parallel^a$ given by ([55])

$$V_\parallel^a = P^a{}_b V^b, \quad V_\perp^a = n^a n_b V^b. \tag{B.4}$$

To see that the first vector is indeed tangent to $\Sigma$, we show that it is perpendicular to $n^a$:

$$\begin{aligned} n_a V_\parallel^a &= n_a (g^a{}_b - s\, n^a n_b V^b) \\ &= n_a V^a - s\, n^2 n_b V^b \\ &= n_a V^a - n_b V^b \\ &= 0. \end{aligned} \tag{B.5}$$

And to see that $V_\perp^a$ is indeed perpendicular to $\Sigma$ we show that it is collinear to $n^a$:

$$\begin{aligned} n_a V_\perp^a &= s\, n_a n^a V^b \\ &= s^2 n_b V^b \\ &= n_b V^b \end{aligned} \tag{B.6}$$

which is the projection of the total vector $V$ on $n$. So the vector given by $P^a{}_b V^b = V_\parallel^a$ is in fact the projection of $V^a$ on $\Sigma$.

The projection tensor actually plays the role of a metric tensor on $\Sigma$, i.e. it is the *induced metric* on $\Sigma$. To see this, let us consider two vectors $V^a, W^a \in T_p(\Sigma)$. The projection tensor raises and lowers the indices:

$$P^{ab} V_b = g^{ab} V_b - s\, n^a n^b V_b. \tag{B.7}$$

Since $V^b$ is normal to all vectors in $\Sigma$, the contraction $n^b V_b$ vanishes, and we are left with

$$P^{ab} V_b = V^b. \tag{B.8}$$

Second, the projection tensor allows us to calculate the scalar product of two vectors in $\Sigma$:

$$\begin{aligned} P_{ab}V^aW^b &= g_{ab}V^aW^b - sn_an_bV^aW^b \\ &= V^aW_a. \end{aligned} \tag{B.9}$$

Finally, the projection tensor is idempotent, meaning that it produces the same result when applied more than once:

$$\begin{aligned} P^a{}_cP^c{}_b &= (\delta^a_c - sn^an_c)(\delta^c_b - sn^cn_b) \\ &= \delta^a_b - sn^an_b - sn^an_b + s^2n^an_b \\ &= P^a{}_b. \end{aligned} \tag{B.10}$$

The projection tensor is sometimes called the *first fundamental form of* $\Sigma$.

Now that we have presented the concept of hypersurfaces, a particular type of hypersurface that we will see throughout the thesis is the so-called $n$-sphere. It is the generalization of the concept of a 2-dimensional spherical surface (a.k.a. a *sphere*) to $n$ dimensions. An $n$-sphere is hence a hypersurface defined as follows:

**Definition B.1.** Let $d : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^+$ be the distance function on $\mathbb{R}^{n+1}$. The $n$-sphere of radius $R$ and center $c$ is defined by

$$S^n(c, R) := \{x \in \mathbb{R}^{n+1} \,|\, d(c, x) = R\}. \tag{B.11}$$

We usually refer to an $n$-spheres in an abstract topological manner without specifying the center and radius, and simply denote it by $S^n$.

It follows that we define the $(n+1)$-ball $B_R(c)$ as the space enclosed by an $n$-sphere. The $(n+1)$-ball $B_R(p)$ is *closed* if it contains the $n$-sphere and *open* if it does not.

## B.2 Extrinsic Curvature

The reader is probably familiar with the notion of curvature on a manifold. Take a vector $V^a \in T_{p_0}(\Sigma)$ and parallel-transport it around a small loop, ending back at the starting point $p$. If the vector is the same then the manifold is flat. If not, then the manifold has curvature. To actually quantify this notion we would pick two vectors $A^a$ and $B^a$ which will define the rotation plane, then argue that the difference in $V^a$ must be proportional to the original vector as well as the two vectors $A^a$ and $B^a$. The proportionality "factor" of course turns out to be the Riemann tensor – which must be a function of the four variables that are the old and the new vectors, $A^a$ and $B^a$. It can hence be expressed as a 4-tensor.

The above procedure explains how the curvature is measured by someone on the manifold $\Sigma$ itself and is thus called the **intrinsic curvature**. If you embed $\Sigma$ in a higher-dimensional manifold $\mathcal{M}$, and do the same procedure partly in $\mathcal{M}$, you get what we call **extrinsic curvature**.

As a clear example, think of a cylinder embedded in $\mathbb{R}^3$. The cylinder of course has zero intrinsic curvature: it is simply a sheet with two of its ends identified. This is actually a consequence of Gass's *theorema egregium* which states that, given two surfaces that are defined using maps having the same domain and codomain, if the two surfaces are isometric[1] then they have the same intrinsic curvature [56]. Imagine starting again with a vector $V^a \in T_p(\Sigma)$ and transporting the vector half-way through the cylinder. You can then "lift" the vector off of the $\Sigma$ plane, parallel-transport it all the way back "above" the point $p$ then "drop" it back on the cylinder at $p$. The resulting vector is actually $-V^a$. The two vectors are clearly different and the cylinder posses an extrinsic curvature *when embedded in* $\mathbb{R}^3$.

The extrinsic curvature is quantified using the extrinsic curvature tensor $K_{ab}$, which is given by ([11])

$$K_{ab} = \frac{1}{2}\mathcal{L}_n P_{ab}, \tag{B.12}$$

where $n$ is the unit normal vector to $\Sigma$, $P_{ab}$ is the projection tensor and $\mathcal{L}_n$ is the Lie derivative in the direction of $n$ (c.f. Appendix A). The extrinsic curvature is hence the rate of change of the projection tensor in the direction of the flow of the normal vector to the hypersurface.

With some straightforward manipulations we can arrive at a useful formula for calculating $K_{ab}$:

$$\begin{aligned} K_{ab} &= \frac{1}{2}P^c{}_a P^d{}_b \,\mathcal{L}_n\, g_{ab} \\ &= P^c{}_a P^d{}_b \nabla_{(c} n_{d)} \\ &= \nabla_{(a} n_{b)} \end{aligned} \tag{B.13}$$

The extrinsic curvature is sometimes called the *second fundamental form of* $\Sigma$.

[1] There exists a distance-preserving bijective function from one surface to the other.

## B.3 Geometry of Foliations and Metric Decomposition

In §B.1 we have discussed the concept of a hypersurface. We will discuss in the present section how a spacetime can be covered by a continuous set of timelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$. This is possible for example for a class of spacetimes called globally hyperbolic spacetimes which we discuss below.

We will start by developing some notions that we will be needing in following parts of this section as well as other parts of the thesis. To warm up, let us recall the simple notion of a *neighborhood* in topology.

**Definition B.2.** Let $E$ be a topological space. A subset $\mathcal{V} \subset E$ is called a **neighborhood of $\boldsymbol{x}$ in $\boldsymbol{E}$** if there exists an open subset $U \subset E$ such that $x \in U \subset \mathcal{V}$ [57].

Next, we turn to the notion of *future-directed causal curves.* Remember that a *causal curve* is simply a timelike or null curve.

**Definition B.3.** A spacetime $\mathcal{M}$ is **time-orientable** if it admits a smooth, nowhere vanishing timelike vector field $\hat{t}^a$. Let $x(\lambda)$ be a causal curve defined on $\mathcal{M}$. For each point on the curve $p \in x(\lambda)$, there is a vector tangent to $x(\lambda)$, defined in the tangent space $T_p(\mathcal{M})$ and given by $x(p)$. Then $x(\lambda)$ is said to be a **future-directed causal curve** if $\forall\, p \in x(\lambda), x^a(p)\hat{t}_a(p) < 0$ [15].

We can now discuss the definition for the future domain of dependence.

**Definition B.4.** Let $(\mathcal{M}, g)$ be a spacetime and $S \subset \mathcal{M}$ such that no two points on $S$ can be connected by a timelike curve (we also say that $S$ is **achronal**). The **future domain of dependence** of $S$, denoted $D^+(S)$ (respectively the **past domain of dependence** of $S$, denoted $D^-(S)$), is the set of all points $p \in \mathcal{M}$ with the property that every past-directed (respectively future-directed) inextendible (i.e. with no endpoints) timelike curve starting at $p$ intersects $S$ [11].

**Definition B.5.** A **Cauchy surface** is a spacelike hypersurface $\Sigma$ of a spacetime $(\mathcal{M}, g)$ such that each inextendible causal curve intersects it once and only once. Equivalently, this means that $\mathcal{M} = D^+(\Sigma) \cup D^-(\Sigma)$.

Not all spacetimes have Cauchy surfaces, though. For example, if we were to remove one point from a Minkowski spacetime, there would not be any Cauchy surfaces in it. We hence present the following classification:

**Definition B.6.** A spacetime is said to be **globally hyperbolic** if it admits a Cauchy surface.

We could say that the region $D^+(S)$ is the region of spacetime in which solutions of hyperbolic partial differential equations are provided by initial values on $\Sigma$ [15].

**Definition B.7.** A **partial Cauchy surface** is a spacelike hypersurface such that each causal curve with no endpoints intersects it *at most* once [15].

The topology of a globally hyperbolic spacetimes admitting a Cauchy surface $\Sigma$ is necessarily $\Sigma \times \mathbb{R}$. It is easy to imagine that any globally hyperbolic spacetime for example can be foliated by a set of continuous hypersurfaces (or *foliations*) $(\Sigma_t)_{t\in\mathbb{R}}$, such that

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t. \tag{B.14}$$

Specifically, let us imagine such a $D$-dimensional spacetime $(\mathcal{M}, g)$, foliated by hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$. This means that there exists a regular (first covariant derivative does not vanish) scalar field $\phi : \mathcal{M} \to \mathbb{R}$ such that each hypersurface $\Sigma_t$ is a level of this field [58],

$$\forall\, t \in \mathbb{R},\, \Sigma_t := \{p \in \mathcal{M},\, \phi(p) = t\}. \tag{B.15}$$

If $\phi$ is in fact regular, it is obvious that the hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ are non-intersecting:

$$\forall\, t_1, t_2 \in \mathbb{R} \,|\, t_1 \neq t_2,\, \Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset. \tag{B.16}$$

It is evident that a spatial point $p$ on a slice is connected to its counterpart on the next surface via a *time flow* vector, which we denote by $\hat{t}^a$. We have not specified what this time flow vector is, but its meaning is obvious: it is a vector field that we construct to measure the flow of time. An obvious choice is to simply let it be proportional to the time coordinate:

$$\hat{t}^a = (1, 0, 0...0)\,, \tag{B.17}$$

If the spacetime is static and all foliations are identical, then the time vector connecting $p(t)$ to its future $p(t + \delta t)$ is obviously orthogonal to each slice $\Sigma_t$, and intersects each point on each slice once and only once.

If we now take a spacetime which is not static (but at least stationary like all GR solutions that we are interested in), then the spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ are no longer identical (though they may remain topologically equivalent, but that is not relevant to our discussion). In fact, even for static spacetimes we can foliate our manifold with non identical timelike hypersurfaces given an appropriate $\phi(p)$. The time flow vector connecting each point to its future is thus not, in general, orthogonal to each slice $\Sigma_t$. Furthermore, regardless of whether or not the spacetime is static, we are free to quantify the "time flow" using any timelike vector field other than (B.17).

In conclusion, there are different ways of defining $\hat{t}^a$ and, in general, this vector field will have normal and tangential components to each slice $\Sigma_t$. The tangential part, denoted $\beta^a$, is called the *shift* vector field. It represents how a spatial point is shifted between slices by following the chosen vector $\hat{t}^a$.

As discussed in §A.1, the future-directed normal vector to each slice, denoted $u^a$, will be collinear to $\nabla^a t$,

$$\exists\, N : \mathcal{M} \to \mathbb{R} \,|\, u(x)^a = -N(x)\nabla^a t,^{2} \tag{B.18}$$

which leads directly to the one-form

$$u = -N dt. \tag{B.19}$$

The scalar field $N$ is called the *lapse function*. We chose the normal component of $\hat{t}$ to be

$$\hat{t}_\perp = Nu. \tag{B.20}$$

On each hypersurface $\Sigma_t$, we introduce a coordinate system

$$(x^i)_i = (x^1, ..., x^{D-1}).$$

If this coordinate system varies smoothly between slices, then

$$(x^\mu)_\mu = (t, x^1, ..., x^{D-1}) \tag{B.21}$$

forms a well-behaved coordinate system on $\mathcal{M}$. We now pick

$$\hat{t} = \partial_t. \tag{B.22}$$

Since $\beta$ is normal to $\Sigma_t$, let us introduce the spatial coordinates of $\beta$ terms of the spatial coordinates $(x^i)$:

$$\beta = \beta^i \partial_i. \tag{B.23}$$

We now turn our attention to the components of the complete spacetime metric. We start by showing that

[2] $N$ is real because in GR we are dealing with Lorentzian manifolds which, by definition, are real manifolds.

$$
\begin{aligned}
g_{00} &= g(\partial_t, \partial_t) \\
&\equiv g_{\mu\nu}(\partial_t)^\mu(\partial_t)^\nu \\
&= -N^2 + \beta^i\beta_i,
\end{aligned} \tag{B.24}
$$

where we used (B.22) in the last equality. The next set of components are those of the form

$$
\begin{aligned}
g_{0i} &= \partial_t \cdot \partial_i \\
&= \beta \cdot \partial_i \\
&= \beta_j \mathrm{d}x^j \partial_i \\
&= \beta_j \delta_i^j \\
&= \beta^i.
\end{aligned} \tag{B.25}
$$

The remaining components are of the form $g_{ij}$. Let us denote the spatial induced metric on each slice by $\sigma_{ij}$. It is the bilinear form defined by

$$
\forall\,(v, w) \in T_p(\Sigma_t) \times T_p(\Sigma_t),\ \sigma(v, w) := g(v, w). \tag{B.26}
$$

So the full metric tensor can be represented by

$$
g_{ab} = \begin{pmatrix} -N^2 + \beta^i\beta_i & \beta_j \\ \beta_i & \sigma_{ij} \end{pmatrix}. \tag{B.27}
$$

Or as a line element [58]:

$$
g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -N^2\mathrm{d}t^2 + \sigma_{ij}(\mathrm{d}x^i + \beta^i\mathrm{d}t)(\mathrm{d}x^j + \beta^j\mathrm{d}t). \tag{B.28}
$$

This means that the foliation metric can be expressed as

$$
\sigma_{ab} = g_{ab} + u_a u_b. \tag{B.29}
$$

# Appendix C

# Symmetries and Killing Vectors

## C.1 Symmetry Transformations

### C.1.1 Symmetries and Conservation Laws

A symmetry is a transformation on the dynamical variables of the system that leaves the equations of motion unchanged. For instance, a perfect sphere remains exactly the same under a rotation transformation around any of its axes with any angle. If we are studying the dynamics of particles that resides on this sphere, the equations of motion would remain identical if we would rotate the sphere, or, equivalently, if we would rotate the test object around the sphere. Physicists have long been interested in symmetries because knowing the symmetries of a problem can reduce its complexity greatly.

Another feature that is important to know in any physical problem is the underlying conservation laws. Conservation laws are fundamentally important because they tell us which processes can occur and which cannot. They are usually expressed in terms of *continuity equations*. For instance, in electromagnetism the continuity equation which expresses charge conservation is given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0, \tag{C.1}$$

where $\rho$ is the volume charge density, and $\vec{J}$ is the current density (per unit time).

We can of course define a four-vector $J^a$ by

$$J^a = (\rho, \vec{J}), \tag{C.2}$$

and re-write the continuity equation as the conservation of the current $J^a$,

$$\nabla_a J^a = 0. \tag{C.3}$$

The charge associated with this current, which is constant in time, is then [36]

$$Q := \int_{\text{all space}} J^0 \mathrm{d}^3 x. \tag{C.4}$$

Conservation laws are related to underlying symmetries via **Nöther's theorem**. Nöther's theorem states that every differentiable symmetry of the action of a theory has a corresponding conservation law.

If we take $\phi$ to be a field, we can write an infinitesimal transformation as

$$\phi \to \phi' = \phi + \Delta\phi, \tag{C.5}$$

where $\Delta\phi$ is some deformation of the field configuration. This transformation is hence a symmetry if the action in invariant under it, which means that the equations of motion remain unchanged. This means that the Lagrangian must be invariant under (C.5), up to a surface term, since this term would not change the Euler-Lagrange equation. In other words, the equations of motion are unchanged if the effect of equation (C.5) on the Lagrangian is a transformation of the form

$$\mathcal{L} \to \mathcal{L}(\phi, \nabla_a \phi) + \nabla_a \mathcal{J}^a. \tag{C.6}$$

Let us compute the change in $\mathcal{L}$ and see if this is the case.

$$\begin{aligned}
\Delta\mathcal{L}(\phi, \nabla_a\phi) &= \frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\nabla_a \phi)}\nabla_a(\Delta\phi) \\
&= \nabla_a\left(\frac{\partial \mathcal{L}}{\partial(\nabla_a\phi)}\Delta\phi\right) + \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial(\nabla_a\phi)}\right)\Delta\phi \\
&= \nabla_a\left(\frac{\partial \mathcal{L}}{\partial(\nabla_a\phi)}\Delta\phi\right).
\end{aligned} \tag{C.7}$$

The second term in the last line vanishes by the Euler-Lagrange equation. Making the identification

$$J^a = \left(\frac{\partial \mathcal{L}}{\partial(\nabla_a \phi)} \Delta\phi\right), \tag{C.8}$$

we see that if the Lagrangian density is invariant under the transformation (C.6), then the current $J$ must be *divergenceless* (we also say that this *current is conserved*) ,

$$\nabla_a J^a = 0. \tag{C.9}$$

This divergenceless current leads to a *conserved charge*. By this we mean a physical quantity that is invariant with time. To see this, we separate the time and space coordinates in the previous equations, arriving at

$$\nabla_0 J^0 = \nabla_i J^i. \tag{C.10}$$

In general, the left-hand side can be a function of time. However, integrating both sides over a region $\Sigma$ in space (where $\Sigma$ can very well be the whole space), the resulting right hand side is simply a surface term that does not depend on time. This means that the left-hand side given by

$$\int_\Sigma \mathrm{d}^3 x J^0 =: Q, \tag{C.11}$$

is constant in time.

So in conclusion, it is easy to see that the existence of a continuous symmetry in a theory leads to the existence of a conserved charge in said theory.

One type of symmetry that we commonly refer to in physics is *gauge symmetry*. A theory with a gauge symmetry is called a *gauge theory*. More fundamentally, a *gauge* is a mathematical artifact used to regulate redundant degrees of freedom in a theory. A typical example is classical electrodynamics. The gauge of this theory is the four-potential $A = (\Phi, \vec{A})$. The fields are given by

$$\begin{aligned} \vec{E} &= -\vec{\nabla} A^0 - \partial_t \vec{A}, \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{aligned} \tag{C.12}$$

It is easy to see that, given any twice-differentiable function $f : \mathbb{R}^{1,3} \to \mathbb{C}$, the transformation

$$A \to A' = A + \left(-\partial_t f, \vec{\nabla} f\right), \tag{C.13}$$

leaves the fields in (C.12) – and hence Maxwell's equations of motion – unchanged. Classical electrodynamics is therefore a *gauge theory*.

**Proof:** Re-writing the new electric field using the transformed four-potential gives

$$\begin{aligned}\vec{E}' &= -\vec{\nabla}A'^0 - \partial_t\vec{A}' \\ &= -\vec{\nabla}\left(A^0 - \partial_t f\right) - \partial_t\vec{A} - \partial_t\vec{\nabla}f \\ &= \vec{E}.\end{aligned} \tag{C.14}$$

Likewise, for an abstract component $B_i$ of the vector $\vec{B}$ (where the cyclic order $(i, j, k)$ is maintained), we have

$$\begin{aligned}B_i' &= \partial_j A_k - \partial_k A_j + \partial_j\partial_k f - \partial_k\partial_j f \\ &= B_i.\end{aligned}$$

□

Another type of symmetry that is often discussed is *conformal symmetry*. A **conformal transformation** is a bijective transformation[1] that locally preserves the angles. This means that the infinitesimal neighborhood around any point is geometrically similar to its transformed counterpart. It is essentially a local re-scaling of the system. Particularly in General Relativity, it can be expressed using a non-vanishing function $\Omega : \mathcal{M} \to \mathbb{R}_+^*$ by the metric transformation

$$g_{ab}(x) \to \Omega(x)^2 g_{ab}, \tag{C.15}$$

where $g_{ab}$ is the metric on the manifold $\mathcal{M}$.

### C.1.2 Symmetry Groups

Let us consider the operator $\pi$ defined by

$$\pi f(x) = f(-x), \forall\, f : \mathbb{R} \to \mathbb{K}, \tag{C.16}$$

where $\mathbb{K}$ is some random set. We can see that

[1] Can be expressed in terms of a bijective function.

$$\pi\pi = \mathbb{1}, \tag{C.17}$$

where $\mathbb{1}$ is the identity operator such that

$$\mathbb{1} f(x) = f(x). \tag{C.18}$$

You can see that any product we form from the two operators $\pi$ and $\mathbb{1}$ belongs to the set $\{\mathbb{1}, \pi\}$. In this sense we say that the set $\{\mathbb{1}, \pi\}$ is **closed** with respect to the operators product.

We can use a number of operators to form a mathematical group. Recall that a group is any set $G$ together with an operation $*$ that satisfy the *group axioms* below [59]:

1. Closure: $\forall\, a, b \in G, a * b \in G$.
2. Associativity: $\forall\, a, b, c \in G, (a * b) * c = a * (b * c)$.
3. Existence of identity element: $\exists\, I_d \in G \,|\, \forall\, a \in G, a * I_d = I_d * a = a$.
4. Existence of inverse elements: $\forall\, a \in G\, \exists\, \alpha \in G \,|\, a * \alpha = \alpha * a = I_d$. We commonly denote $\alpha$ by $a^{-1}$.

The number of elements $g$ of $G$ is called the **order of the group**. Of course, depending on the group, $g$ can be infinite.

It is easy to see that the set $\{\mathbb{1}, \pi\}$ forms a group, which we denote by $G_1$. This group obeys the *multiplication table* below:

| | |
|---|---|
| $\mathbb{1}$ | $\pi$ |
| $\pi$ | $\mathbb{1}$ |

This multiplication table is formed such that the product of any two elements is the intersection of the row of the first element with the column of the second element.

Now consider another set, that of the integers $\{1, -1\}$. Under ordinary scalar multiplication, these form a group $G_2$. This group has similar properties to the previous one. If we let $a$ be either $-1$ or $\pi$ and $I_d$ be either $\mathbb{1}$ or 1, both groups obey the multiplication table given by

| | |
|---|---|
| $I_d$ | $a$ |
| $a$ | $I_d$ |

This table defines the *abstract group* $C_i$. This abstract group has many realizations or **representations**. These representations are concrete groups that are isomorphic to each other. By isomorphic we mean that there exists a one-to-one correspondence between the distinct elements of the groups.

One interesting property to look at for the elements of a group is the commutativity of its elements. Commutative groups are called **Abelian** and non-commutative groups are called **non-Abelian**. In the previous example $C_i$ was an Abelian group, and we were able to find a representation of this group using numbers (i.e. $G_2$). This is normal since ordinary numbers commute. To find representations for non-Abelian groups on the other hand, we need to use matrices which have non-commutative properties.

There is a number of interesting non-Abelian groups that we will very briefly discuss. First, the *orthogonal group of degree* $N$, denoted $\mathrm{O}(N)$, is the group of orthogonal $N \times N$ matrices together with the matrix multiplication operation. An orthogonal matrix $O$ is a square matrix whose inverse is equal to its transpose,

$$O^T O = O \left(O^T\right) = \mathbb{1}. \tag{C.19}$$

It has a subgroup denoted $\mathrm{SO}(N)$ and called the *special orthogonal group*. It is defined via the subset of matrices with determinant equal to $+1$. It is useful to know that all possible rotation transformations (as expressed using rotation matrices) in $\mathbb{R}^3$ form a representation of the group $\mathrm{SO}(3)$.

Another group that we will mention quickly is the group of unitary $N \times N$ matrices denoted $\mathrm{U}(N)$. A unitary matrix $U$ is one whose conjugate transpose is equal to its inverse,

$$U^\dagger = U^{-1}. \tag{C.20}$$

$\mathrm{U}(N)$ has an interesting subgroup called the *special unitary group of degree* $N$ and denoted $\mathrm{SU}(N)$. It is formed by the matrix product and the set of unitary $N \times N$ matrices in $\mathrm{U}(N)$ with determinant $+1$.

When the symmetry transformations of a particular physical system form a group, we call this a **symmetry group**. For instance, we mentioned above that the set of rotation transformations in $\mathbb{R}^3$ form the group $\mathrm{SO}(3)$. Hence, $\mathrm{SO}(3)$ is the symmetry group of a sphere in ordinary 3-dimensional space. I.e., the sphere remains unchanged under any and all transformations of the group $\mathrm{SO}(3)$. A more advanced example is to consider the quantum field theory that governs the strong interaction, *quantum chromodynamics* (QCD). QCD turns out to have an $\mathrm{SU}(3)$ symmetry group [3].

## C.2 Stokes' Theorem and Conserved Charges in Curved Spacetimes

Stokes' theorem is one of the most important results in Differential Geometry. It generalizes an important result from calculus [11]:

$$\int_b^a \mathrm{d}x = a - b. \tag{C.21}$$

Take a $(D-1)$-form $\omega$, defined on the boundary $\partial\mathcal{M}$ of a $D$-dimensional region $\mathcal{M}$. Then $\mathrm{d}\omega$ is a $D$-form and can be integrated over $\mathcal{M}$. Stokes' theorem says that

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{C.22}$$

This general form leads to several familiar theorems in 3-dimensions calculus, like the theorems of Ampère and Green-Ostrogradski.

To see how this theorem can be put to practical use in calculations, we consider a one-form $J$ defined by

$$\omega = \star J. \tag{C.23}$$

The exterior derivative of $\omega$ in terms of the components of $J$ is then

$$\begin{aligned}(\mathrm{d}\omega)_{ca_1 \ldots a_{D-1}} &= (\mathrm{d} \star J)_{ca_1 \ldots a_{D-1}} \\ &= D\, \epsilon^b{}_{[a_1 \ldots a_{D-1}} \nabla_{c]} J_b.\end{aligned} \tag{C.24}$$

Now, recall that the Levi-Civita tensor is the volume element, i.e.

$$\begin{aligned}\epsilon &= \sqrt{|g|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \\ &\equiv \sqrt{|g|}\, \mathrm{d}^D x.\end{aligned} \tag{C.25}$$

This leads to

$$\mathrm{d}\omega = \nabla_a J^a \sqrt{|g|}\, \mathrm{d}^D x. \tag{C.26}$$

Having found a practical expression for $\mathrm{d}\omega$, we now look for one for $\omega$. Expanding the expression in (C.23), we have

$$\omega = \epsilon_{ba_1 \ldots a_{D-1}} J^b \tag{C.27}$$

Evidently, the induced volume element on a hypersurface (in this case the boundary $\partial\mathcal{M}$) is

$$\hat{\epsilon} = \sqrt{h}\, \mathrm{d}^{D-1} y. \tag{C.28}$$

Here $h_{ab}$ is the induced boundary metric in coordinates $y^a$. The induced volume form in terms of the volume form in $\mathcal{M}$ can be found my simply contracting the later with the unit normal vector,

$$\hat{\epsilon}_{a_1 \ldots a_{D-1}} = n^c \epsilon_{ca_1 \ldots a_{D-1}}, \tag{C.29}$$

where $n^a$ is the unit normal vector to the boundary hypersurface, and $\hat{\epsilon}$ is the volume form on the latter. Combining this with (C.23), we get

$$\omega = n_a V^a \sqrt{|h|}\, \mathrm{d}^{D-1} y. \tag{C.30}$$

Combining (C.26) with (C.30), we can re-write (C.22) as

$$\boxed{\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|g|}\, \nabla_a J^a = \int_{\partial\mathcal{M}} d^{D-1} y \sqrt{|h|}\, n_a J^a.} \tag{C.31}$$

That is it, we have arrived at a practical version of Stokes' theorem which can easily be used in our calculations.

Now let us recall that a current $J^a$ is conserved if

$$\nabla_a J^a = 0. \tag{C.9}$$

From equation (C.26), we see that $\mathrm{d} \star J$ is proportional to $\nabla_\mu J^\mu$, and hence the current conservation condition can also be expressed as

$$\mathrm{d}(\star J) = 0 \Rightarrow J \text{ is a conserved current.} \tag{C.32}$$

Equation (C.32) is an important result which will be referenced several times in this thesis. In the spirit of the previous section, we can now define a conserved charge passing through a hypersurface $\Sigma$ using

$$\boxed{Q_\Sigma = \int_\Sigma \star J.} \tag{C.33}$$

From the previous calculations, we can rewrite $Q_\Sigma$ as

$$Q_\Sigma = \int_\Sigma \mathrm{d}^{D-1}y\,\sqrt{|h|}\,n_a J^a. \tag{C.34}$$

Let us pause to recap what we have done so far. We have found that a current $J^a$ is conserved if $\mathrm{d}\star J = 0$. We have also found a form of Stokes' theorem which we claimed will be useful in our calculations. To see how all this can be used in the real world, let us calculate the electric charge resulting from the conserved electromagnetic current $J^\mu = \nabla_\nu F^{\mu\nu}$. Using C.31 we can write

$$\int_{\Sigma_t} \mathrm{d}^{D-1}y\,\sqrt{|h|}\,n_b \nabla_a F^{ab} = -\int_{\partial\Sigma_t} \mathrm{d}^{D-2}z\,\sqrt{|\gamma|}\,u_a\,n_b F^{ab}, \tag{C.35}$$

where $\Sigma_t$ is a spatial hypersurface (at constant $t$), $h_{ab}$ is the induced metric on $\Sigma_t$, $u$ and $n$ are the unit normals to $\Sigma_t$ and $\partial\Sigma_t$, respectively, and $z^a$ and $\gamma_{ab}$ are the coordinates and induced metric on $\partial\Sigma_t$. To compare this with the results from the previous section, we take the 4-dimensions, flat spacetime case. $\Sigma_t$ can then be taken as the 3-ball $B_R(p) = \{x \in \mathbb{R}^3 \,|\, d(x,p) < R\}$, where $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$ is again the distance function on $\mathbb{R}$. $\partial B_R$ will of course be the familiar 2-sphere (denoted $S^2$). Working in spherical coordinates, the metrics are given by

$$h_{ij}\mathrm{d}y^i\,\mathrm{d}y^j = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\phi^2, \tag{C.36}$$

and

$$\gamma_{ij}\mathrm{d}z^i\,\mathrm{d}z^j = r^2\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\phi^2. \tag{C.37}$$

The electric field has one non-vanishing component

$$E^r = \frac{q}{4\pi r^2}, \tag{C.38}$$

and the field-strength tensor has the component

$$F^{tr} = E^r. \tag{C.39}$$

The one-forms associated with the normal unit vectors are easily found to be

$$u_a = (1\ \ 0\ \ 0\ \ 0), \tag{C.40}$$

and

$$n_a = (0\ \ 1\ \ 0\ \ 0). \tag{C.41}$$

Putting it all together, we have

$$Q = -\int_{S^2} \mathrm{d}\theta\, \mathrm{d}\phi\, R^2 \sin^2\theta \left(-\frac{q}{4\pi R^2}\right) = q. \tag{C.42}$$

This is the familiar answer that we were expecting to retrieve. In Chapter 1 we will see how this formalism can be used to calculate charges associated with black holes. While it is obvious that here we could have done without all the complicated results derived in this section, when we are talking about a black hole, we have to take into consideration the curvature of the spacetime and these results become imperative.

## C.3 Killing Vectors

A spacetime $(\mathcal{M}, g)$ has a symmetry if its geometry is invariant under a certain transformation. These symmetries of the metric are called **isometries**. A simple of example of an isometry is the translation transformation in Minkowski space:

$$x^a \to x^a + b^a. \tag{C.43}$$

Whenever a metric is independent of a specific fixed coordinate $\alpha$, there will be an isometry under translation in this coordinate:

$$\partial_\alpha g_{ab} = 0 \Rightarrow x^\alpha \to x^\alpha + b^a \quad \text{is a symmetry.} \tag{C.44}$$

Now, recall that the geodesic equation for timelike curves can be written in terms of the four-momentum $p^a = mU^a$ as

$$p^b \nabla_b p^a = 0. \tag{C.45}$$

Expanding the above expression, we arrive at

$$p^b \partial_b p_a - \Gamma^c_{ba} p^b p_c = 0. \tag{C.46}$$

The first is term simply

$$\begin{aligned} p^b \partial_b p_a &= m \frac{dx^b}{d\tau} \partial_b p_a \\ &= m \frac{dp_a}{d\tau}. \end{aligned} \tag{C.47}$$

And with some straightforward calculations, we find the second term to be

$$\begin{aligned} \Gamma^c_{da} p^d p_c &= \frac{1}{2} g^{cb} \left( \partial_d g_{ab} + \partial_a g_{bd} - \partial_b g_{da} \right) p^d p_c \\ &= \frac{1}{2} \left( \partial_d g_{ab} + \partial_a g_{bd} - \partial_b g_{da} \right) p^d p^b \\ &= \frac{1}{2} \left( \partial_a g_{bd} \right) p^d p^b. \end{aligned} \tag{C.48}$$

Which leads to the equation

$$m \frac{\mathrm{d}p_a}{\mathrm{d}\tau} = \frac{1}{2} (\partial_a g_{bc}) p^c p^b. \tag{C.49}$$

This means that if all metric components are independent of a coordinate $\alpha$ (meaning $\partial_\alpha g_{ab} = 0$), then the momentum component $p_\alpha$ is a conserved quantity of motion, since from (C.49) we will have

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = 0. \tag{C.50}$$

Although metric independence from a specific component implies an isometry, not all metric isometries are related to coordinate independence. For example, the same Minkowski space is also invariant under Lorentz transformations ($x^a \to \Lambda^a{}_b\, x^b$). This set of isometries is not manifest in the independence of the metric $\eta_{ab}$ on any coordinates. In fact, certain coordinate transformations could lead us to write the Minkowskian metric in a such a way that no coordinates independence exists. Since the geometry itself did not change, there is still an isometry related to translation but we would not be able to identify it simply by looking at the metric. It is obvious that a more ingrained practice needs to be developed to find all the underlying symmetries.

Assuming the metric is independent of a coordinate $\alpha$, let us consider the vector

$$K^b = (\partial_\alpha)^b. \tag{C.51}$$

$K^b$ is said to be a *generator of the isometry*. To see this, first note that the component $p_\alpha$ can be written as

$$p_\alpha = K_b p^b. \tag{C.52}$$

Plugging this in the geodesic equation, we have the conclusion

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = 0 \;\Leftrightarrow\; p^a \nabla_a (K_b p^b) = 0. \tag{C.53}$$

Using the geodesic equation, the expression on the right can be expanded as

$$\begin{aligned} p^a \nabla_a (K_b p^b) &= p^a p^b \nabla_a K_b \\ &= p^a p^b \nabla_{(a} K_{b)}. \end{aligned} \tag{C.54}$$

In the last line, we have used the fact that $p^a p^b$ is symmetric in $a$ and $b$, and thus, only the symmetric part of the tensor $\nabla_a K_b$ survives. Consequently, if any vector $K$ satisfies the equation

$$\boxed{\nabla_{(a} K_{b)} = 0}, \tag{C.55}$$

then the quantity $K^a p_a$ is conserved along a geodesic trajectory. Equation (C.55) is called **Killing's equation** and any vector that satisfies it is called a **Killing vector**. As we will see, Killing vectors play a pivotal role in defining conserved charges associated with the isometries of the spacetime.

We would now like to look at some interesting Killing vector identities that we will be making use of in the thesis. The first identity is

$$\nabla_a \nabla_c K^d = R^d{}_{cab} K^b. \tag{C.56}$$

**Proof:** Applying a covariant derivative to Killing's equation gives

$$\nabla_a \nabla_b K_c + \nabla_a \nabla_c K_b = 0. \tag{C.57}$$

Obviously this equation holds after re-labeling of the indices. We can hence do some permutations and add four copies of the right-hand side to itself. If we are clever about the signs of each copy, we can write

$$\begin{aligned} 0 =& [\nabla_a \nabla_b K_c + \nabla_a \nabla_c K_b] - [\nabla_c \nabla_a K_b + \nabla_c \nabla_b K_a] &\text{(C.58)}\\ &+ [\nabla_b \nabla_c K_a + \nabla_b \nabla_a K_c] - [\nabla_a \nabla_b K_c + \nabla_a \nabla_c K_b]. &\text{(C.59)}\end{aligned}$$

It is easy to see that several terms in this expression will cancel (for example the second and third terms). We can then combine some terms as commutators,

$$0 = \nabla_a \nabla_b K_c - \nabla_a \nabla_c K_b + [\nabla_a, \nabla_c] K_b - [\nabla_c, \nabla_b] K_a + [\nabla_b, \nabla_a] K_c. \tag{C.60}$$

The commutator of two covariant derivatives applied to any vector $V$ can, in general, be expressed as

$$\begin{aligned} [\nabla_a, \nabla_b] V^d &= \nabla_a \nabla_b V^d - \nabla_b \nabla_a V^d \\ &= \partial_a \left(\nabla_b V^d\right) - \Gamma^e_{ab} \nabla_e V^d + \Gamma^d_{ae} \nabla_b V^e - \partial_b \left(\nabla_a V^c\right) + \Gamma^e_{ba} \nabla_e V^d - \Gamma^d_{be} \nabla_a V^e \\ &= \partial_a \partial_b V^d + \partial_a \left(\Gamma^d_{be} V^e\right) + \Gamma^d_{ae} \partial_b V^e + \Gamma^d_{ae} \Gamma^e_{bf} V^f - \partial_b \partial_a V^d - \partial_b \left(\Gamma^c_{ae} V^e\right) \\ &\quad - \Gamma^d_{be} \partial_a V^e - \Gamma^d_{be} \Gamma^e_{af} V^f \\ &= \left(\partial_a \Gamma^d_{bf} - \partial_b \Gamma^d_{af} + \Gamma^d_{ae} \Gamma^e_{bf} - \Gamma^d_{be} \Gamma^e_{af}\right) V^f \\ &= R_d{}^{fab} V^f. \end{aligned} \tag{C.61}$$

Using this in equation (C.60), we get

$$\nabla_a \nabla_c K_d = \frac{1}{2} \left([\nabla_a, \nabla_c] K_d [\nabla_d, \nabla_c] K_a + [\nabla_d, \nabla_a] K_c\right). \tag{C.62}$$

Using the fact that the sum of cyclic permutations of the form $R_{dbac} + R_{dacb} + R_{dcba}$ vanishes, we can combine the first and last Riemann tensor from above to get

$$\begin{aligned}
\nabla_a \nabla_c K_d &= \frac{1}{2} \left( R_{abdc} - R_{dcba} \right) K^b \\
&= \frac{1}{2} \left( R_{abdc} + R_{abdc} \right) K^b \\
&= R_{dcab} K^b.
\end{aligned} \tag{C.63}$$

Multiplying by the inverse metric tensor leads to the final result

$$\nabla_a \nabla_c K^d = R^d{}_{cab} K^b. \tag{C.64}$$

□

Contracting relation (C.56), it is easy to arrive at another important result,

$$\nabla_a \nabla_c K^a = R_{cb} K^b. \tag{C.65}$$

Equations (C.65) also leads to another important result, namely that the directional derivative of the Ricci scalar along any Killing vector vanishes,

$$K^a \nabla_a R = 0. \tag{C.66}$$

**Proof:** We apply $\nabla_b$ to equation (C.65),

$$\nabla_b \nabla_a \nabla_c K^a = K^b \nabla_b R_{cb} + R_{cb} \left( \nabla_b K^b \right). \tag{C.67}$$

Then, raising the $c$ index,

$$\nabla_b \nabla_a \nabla^c K^a = K^b \nabla_b R^c{}_b + R^c{}_b \nabla_b K^b. \tag{C.68}$$

Setting $c = b$,

$$\nabla_b \nabla_a \nabla^b K^a = K^b \nabla_b R + R \nabla_b K^b. \tag{C.69}$$

Looking at the last term on the right, it is easy to see that it vanishes:

$$
\begin{aligned}
R\nabla_b K^b &= R\nabla_b (g^{ba} K_a) \\
&= R g^{ba} \nabla_b K_a \\
&= 0.
\end{aligned} \tag{C.70}
$$

In the second line I have used metric compatibility, and in the last line I have used the fact that a symmetric tensor (the metric) is contracted with an anti-symmetric tensor. So we are left with

$$
\begin{aligned}
K^b \nabla_b R &= \nabla_b \nabla_a \nabla^b K^a \\
&= -\nabla_b \nabla_a \nabla^a K^b.
\end{aligned} \tag{C.71}
$$

In local inertial coordinates, the left-hand side can be written as:

$$K^b \nabla_b R \mathrel{\hat{=}} -\partial_b \Box K^b \tag{C.72}$$

$$= -\Box \partial_b K^b \tag{C.73}$$

$$= -\Box g^{ba} \partial_b K_a, \tag{C.74}$$

where again we use the symbol "$\hat{=}$" to mean "equal in a specific frame". The term $\partial_b K_a$ is not (i.e. does not transform as) a tensor. However, we can always write it as the sum of

$$\partial_{(a} K_{b)} = \frac{1}{2} \left( \partial_a K_b + \partial_b K_a \right), \tag{C.75}$$

and

$$\partial_{[a} K_{b]} = \frac{1}{2} \left( \partial_a K_b - \partial_b K_a \right). \tag{C.76}$$

And using Killing's equation, we know that only the last term is non-zero, hence

$$K^b \nabla_b R \mathrel{\hat{=}} -\Box g^{ba} \partial_{[a} K_{b]}. \tag{C.77}$$

The term $\partial_{[a} K_{b]}$ does transform like a tensor. To see this, notice that

$$
\begin{aligned}
\nabla_a K_b &= \partial_a K_b - \Gamma^{\lambda}_{ab} K_{\lambda}, \\
\nabla_b K_a &= \partial_b K_a - \Gamma^{\lambda}_{ba} K_{\lambda}.
\end{aligned} \tag{C.78}
$$

Using torsion freedom, we can see that

$$
\nabla_a K_b - \nabla_b K_a = \partial_a K_b - \partial_b K_a. \tag{C.79}
$$

And hence, the quantity $\partial_{[a} K_{b]}$ does transform like a tensor. Since it is anti-symmetric, its contraction with $g^{ab}$ leads to zero, and we arrive at

$$
K^b \nabla_b R = 0. \tag{C.80}
$$

Since this is a tensor equation, it is true in any coordinate frame.

□

While the existence of a Killing vector allows us to find a conserved quantity for the motion of particles, it also allows us to define a conserved current by

$$
J^a_T = K_b T^{ab}. \tag{C.81}
$$

It is easy to show that this current is divergenceless:

$$
\nabla_a J^a_T = (\nabla_a K_b) T^{ab} - K_b \nabla_a T^{ab}. \tag{C.82}
$$

The first term vanishes because again we have a contraction between an antisymmetric and a symmetric tensor. The second term of course vanishes because of energy-momentum conservation.

We know from §C.2 that conserved currents of the form (C.81) lead to conserved charges of the form (C.34). Therefore, the existence of a Killing vector $K$ allows us to define an associated conserved charge

$$
Q_K = \int_{\Sigma} \mathrm{d}^{D-1} x \sqrt{|h|} J^a_T n_a. \tag{C.83}
$$

This underlying relation between Killing vectors and conserved charges will be used to defined conserved quantities in the black hole solution that are otherwise very ambiguous to define.

# Appendix D

# Action Calculation and the Energy-Momentum Tensor

Famous theoretical physicist Leonard Susskind once said that attempting to derive the Einstein field equation from the action principle is something that he started several times but never actually finished because it is “too tedious” [60]. In the next section we are going to find out why.

## D.1 Action and Energy-Momentum Tensor Definitions

The principle of least action is one of the most essential concepts in physics. All of the systems that we know in classical mechanics, electrodynamics and quantum field theory obey the principle of least action [60]. Meanwhile, the bedrock of general relativity is of course the Einstein field equation, which relates the geometry of the spacetime to the matter distribution in it [11]. The equation in vacuum is given by

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0. \tag{D.1}$$

As is the case in any sensible theory, we expect the above equation of motion of general relativity to result from the principle of least action. This means that we expect to write down an action that, when made stationary, leads to equation (D.1). The bare version of this action turns out to be the sum of two parts. The first is called the **Einstein-Hilbert action** and is given by

$$\boxed{I_{\text{EH}} = -\frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}(R - 2\Lambda)}, \tag{D.2}$$

where $D$ is the dimension of our spacetime which we denote by $(\mathcal{M}, g)$, and the minus sign comes form following the convention in [4]. The second part of the action is called the **York-Gibbons-Hawking action**. It is common to refer to this part simply as the **Gibbons-Hawking action**, which we will do throughout the thesis for simplicity. The Gibbons-Hawking action is given by

$$\boxed{I_{\text{GH}} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x\sqrt{-h}K}, \tag{D.3}$$

where $\partial\mathcal{M}$ is the spatial boundary of $\mathcal{M}$, $h$ is the determinant of the induced boundary metric, and $K$ is the extrinsic curvature tensor of the boundary (c.f. §B.2). In asymptotically flat spacetimes then, the action is simply given by the Einstein-Hilbert term, and in many introductory textbooks it is common to neglect mentioning the Gibbons-Hawking action term. Nevertheless, we will now show that the variation of the total action leads to the field equation in (D.1).

### Variation of the Einstein-Hilbert Action

We now consider variations in the action due to arbitrary variations in the metric $g_{ab}$.

$$\delta I_{\text{EH}} = -\frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^D \left[(R - 2\Lambda)\delta\sqrt{-g} + \sqrt{-g}\delta R\right]. \tag{D.4}$$

It is easy to see that the variation of $g_{ab}$ in terms of the inverse metric $g^{cd}$ is

$$\delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}. \tag{D.5}$$

The reader may recall that for any square matrix $M$ with a non-zero determinant, we have the following identity [11]

$$\ln(\det M) = \mathrm{Tr}(\ln M). \tag{D.6}$$

The variation of this identity then yields

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M). \tag{D.7}$$

This allows us to find the variation of $g$:

$$\delta g = g\, g^{ab}\delta g_{ab}. \tag{D.8}$$

It is sometimes convenient to vary with respect to the inverse metric $g^{ab}$, so we get

$$\delta g = -g\, g_{ab}\delta g^{ab}. \tag{D.9}$$

Using equation (D.5),

$$\begin{aligned}\delta\sqrt{-g} &= \frac{1}{2\sqrt{-g}}\delta(-g)\\ &= -\frac{1}{2}\sqrt{-g}\, g_{ab}\delta g^{ab}.\end{aligned} \tag{D.10}$$

Next we turn our attention to the variation of the Ricci scalar. The Ricci scalar is defined by (1.7). Its variation is then given by two terms:

$$\delta R = (\delta g^{ab})R_{ab} + g^{ab}\delta R_{ab}. \tag{D.11}$$

We can now write the variation of the Einstein-Hilbert term as

$$\begin{aligned}\delta I_{\mathrm{EH}} &= -\frac{1}{16\pi}\left[\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\left(R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab}\right)\delta g^{ab} + \int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}g^{ab}\delta R_{ab}\right]\\ &= -\frac{1}{16\pi}\left[\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\left(G_{ab} + \Lambda g_{ab}\right)\delta g^{ab} + \int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}g^{ab}\delta R_{ab}\right].\end{aligned} \tag{D.12}$$

The term inside the parentheses seems to be on the right track to giving us the sought-after Einstein equation.

For the variation of the Ricci scalar in the second term we refer to the Palatini identity:

$$\delta R_{ab} = \nabla_c(\delta\Gamma^c_{ba}) - \nabla_b(\delta\Gamma^c_{ca}). \tag{D.13}$$

**Proof of the Palatini Identity:** We start by looking at the variation of the Riemann tensor:

$$\delta R^{d}{}_{cab} = \delta\partial_a\Gamma^d_{cb} + (\delta\Gamma^e_{bc})\Gamma^d_{ae} + \Gamma^e_{bc}(\delta\Gamma^d_{ae}) - (a \leftrightarrow b). \tag{D.14}$$

Next we make use of a helpful remark [11]. Imagine defining two covariant derivatives $\nabla$ and $\hat{\nabla}$ with associated connection coefficients $\Gamma$ and $\hat{\Gamma}$. Then for an arbitrary vector $V$,

$$\begin{aligned}(\nabla_a - \hat{\nabla}_a)V^b &= \partial_a V^b + \Gamma^b_{ac}V^c - \partial_a V^b - \hat{\Gamma}^b_{ac}V^c \\ &= (\Gamma^b_{ac}V^c - \hat{\Gamma}^b_{ac})V^c.\end{aligned} \tag{D.15}$$

Since the left-hand side is a tensor, so must be the right-hand side. The difference between the two connection coefficients must then be a tensor. From this, we can infer that the variation $\delta\Gamma^d_{cb}$ must also be a tensor. We can therefore write its covariant derivative as

$$\nabla_a(\delta\Gamma^d_{cb}) = \partial_a(\delta\Gamma^d_{cb}) + \Gamma^d_{ea}\delta\Gamma^e_{cb} - \Gamma^e_{ac}\delta\Gamma^d_{eb} - \Gamma^e_{ab}\delta\Gamma^d_{ce}. \tag{D.16}$$

By inspection with (D.14), we have

$$\delta R^{d}{}_{cab} = \nabla_a(\delta\Gamma^d_{cb}) - \nabla_b(\delta\Gamma^d_{ca}). \tag{D.17}$$

Contracting the above equation, we finally arrive at the Palatini identity,

$$\begin{aligned}\delta R_{cb} &= \delta R^{d}{}_{cdb} \\ &= \nabla_d(\delta\Gamma^d_{cb}) - \nabla_b(\delta\Gamma^d_{cd}).\end{aligned} \tag{D.18}$$

□

Using metric compatibility, the term $g^{ab}\delta R_{ab}$ can be written as

$$\begin{aligned}g^{ab}\delta R_{ab} &= \nabla_d(g^{ab}\delta\Gamma^d_{ab} - g^{ad}\delta\Gamma^d_{ad}) \\ &=: \nabla_d V^d.\end{aligned} \tag{D.19}$$

So the second term in (D.12) can be written as

$$\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-g}\, \nabla_d V^d. \tag{D.20}$$

Using Stokes' theorem in the form (C.31), this can be recast as

$$\begin{aligned}\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-g}\, \nabla_d V^d &= \int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x \sqrt{-h}\, n_a V^a \\ &= \int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x \sqrt{-h}\, n_d \left( g^{ab} \delta\Gamma^d_{ab} - g^{ad} \delta\Gamma^d_{ad} \right),\end{aligned} \tag{D.21}$$

where $n^a$ is the outward unit normal to the boundary $\partial\mathcal{M}$. As in any variational procedure, we demand that the dynamic variable (in this case $g_{ab}$) be fixed at the boundary. Then, given the definition of the Christoffel symbol in (1.1), it is easy to show that

$$\begin{aligned} g^{ad}\delta\Gamma^b_{ab}\big|_{\partial\mathcal{M}} &= \frac{1}{2} g^{ad} g^{bc} (\delta\partial_b g_{ca} + \delta\partial_a g_{cb} - \partial_c \delta g_{ab}) \\ &= \frac{1}{2} g^{dc} g^{ab} (\delta\partial_b g_{ca} + \delta\partial_c g_{ab} - \delta\partial_a g_{cb}). \end{aligned} \tag{D.22}$$

Now we contract this with $n_d$,

$$\begin{aligned} n_d \delta V^d\big|_{\partial\mathcal{M}} &= n^c g^{ab} (\delta\partial_a g_{cb} - \delta\partial_c g_{ab}) \\ &= n^c (n^a n^b + h^{ab}) (\delta\partial_a g_{cb} - \delta\partial_c g_{ab}) \\ &= n^c h^{ab} (\delta\partial_a g_{cb} - \delta\partial_c g_{ab}). \end{aligned} \tag{D.23}$$

In the second line I have used the definition of the projection tensor in equation (B.3). The projection tensor $h^{ab}$ will "project" $\partial g$ on the boundary, making it a tangential derivative. Since we demand that the variation of the metric itself vanish on the boundary, the variation of its tangential derivative must also vanish on the boundary. We are then left with

$$n_d \delta V^d\big|_{\partial\mathcal{M}} = -n^c h^{ab} \delta\partial_c g_{ab}. \tag{D.24}$$

The total variation of the Einstein-Hilbert term is now given by

$$\delta I_{\mathrm{EH}} = -\frac{1}{16\pi} \left[ \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-g}\, (G_{ab} + \Lambda g_{ab})\, \delta g^{ab} - \int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x \sqrt{-h} n^c h^{ab} \delta\partial_c g_{ab} \right]. \tag{D.25}$$

**Variation of the Gibbons-Hawking-York Action**

The extrinsic curvature is given by (B.13), leading to the trace

$$
\begin{aligned}
K &= \nabla_a n^a \\
&= g^{ab}\nabla_b n_a \\
&= (h^{ab} + n^a n^b)\nabla_b n_a \\
&= (h^{ab} + n^b n^a)\nabla_b n_a. \qquad \text{(D.26)}
\end{aligned}
$$

The last term in the last line gives

$$
\begin{aligned}
n^b n^a \nabla_b n_a &= n^b \frac{1}{2}\left(n^a \nabla_b n_a + n^a \nabla_b n_a\right) \\
&= \frac{1}{2} n^b \left(n^a \nabla_b n_a + n_a \nabla_b n^a\right) \\
&= \frac{1}{2} n^b \nabla_b \left(n^a n_a\right) \\
&= \frac{1}{2} n^b \nabla_b (1) \\
&= 0. \qquad \text{(D.27)}
\end{aligned}
$$

So the trace of the extrinsic curvature can be written as

$$
\begin{aligned}
K &= h^{ab}\nabla_b n_a \qquad \text{(D.28)} \\
&= h^{ab}(\partial_b n_a - \Gamma^l_{ab} n_l). \qquad \text{(D.29)}
\end{aligned}
$$

We now need to calculate $\delta K$. First, notice that the variation of $h^{ab}$ is null since we demand that the boundaries (which are analogous to the endpoints in the variational problem in classical mechanics) be fixed. The variation of the trace of the extrinsic curvature can therefore be simplified,

$$
\begin{aligned}
\delta K &= -h^{ab}\delta\Gamma^{l}_{ab}n_l \\
&= -h^{ab}n_l\frac{1}{2}g^{lc}\left(\delta\partial_b g_{ca} + \delta\partial_a g_{cb} - \delta\partial_c g_{ab}\right) \\
&= -\frac{1}{2}h^{ab}\left(\delta\partial_b g_{da} + \delta\partial_a g_{db} - \delta\partial_d g_{ab}\right)n^d \\
&= \frac{1}{2}h^{ab}(\delta\partial_d g_{ab})n^d.
\end{aligned} \tag{D.30}
$$

This term exactly cancels the one in (D.24), when it is divided by $8\pi$ and the latter is divided by $16\pi$. So the variation of the total action $I = I_{\mathrm{EH}} + I_{\mathrm{GH}}$ is given by

$$
\frac{\delta I}{\delta g^{ab}} = -\frac{1}{16\pi}\left[\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\left(G_{ab} + \Lambda g_{ab}\right)\right] \tag{D.31}
$$

The metric is stationary for the values of the field making $\delta I/\delta g^{ab} = 0$, leading to the Einstein equation in vacuum:

$$
\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta g^{ab}} = G_{ab} + \Lambda g_{ab} = 0.
$$

The factor of $\frac{-1}{16\pi}$ is of course irrelevant here since the rightmost side is 0. In the presence of matter, the action gets an extra contribution from the matter fields which we denote $-I_M$. The above equation then becomes

$$
\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta g^{ab}} = -\frac{1}{16\pi}(G_{ab} + \Lambda g_{ab}) - \frac{1}{\sqrt{g}}\frac{\delta I_M}{\delta g^{ab}} = 0. \tag{D.32}
$$

We now define the energy-momentum tensor of any matter fields with action $I_M$ by

$$
T_{ab} = 2\frac{1}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{ab}}, \tag{D.33}
$$

which in turn allows us to recover the Einstein field equation in the presence of matter:

$$
R_{ab} - \frac{1}{2} + \Lambda g_{ab} = 8\pi T_{ab}. \tag{D.34}
$$

## D.2 Action Calculation

We now turn our attention to a particular problem that arises when we try to calculate the action, particularly when we try to perform the integration in (D.2). The issue stems from the fact that a black hole horizon constitutes a coordinates singularity.

Let us take the Schwarzschild metric with arbitrary cosmological constant. We can restrict our discussion here to four- and five-dimensional cases since we will be interested in these. The metric is given by

$$ds^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{D-2}, \tag{D.35}$$

where the form of the ansatz function $f(r)$ depends on $\Lambda$ and the spacetime dimension $D$. One can make a coordinate transformation to an imaginary time by $t \to i\tau$, which leaves the metric looking like

$$ds^2 = f(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{D-2}, \tag{D.36}$$

Because of the signature of (D.36), it is common to refer to such metrics as Euclidean even though they may not be flat. Focusing on the region just outside the black hole, we can expand the metric as

$$ds^2 \approx f'(r_+)(r-r_+)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f'(r_+)(r-r_+)} + r_+^2\mathrm{d}\Omega^2_{D-2}. \tag{D.37}$$

We then define a coordinate $R$ by

$$R = 4\frac{(r-r_+)}{f'(r_+)}, \tag{D.38}$$

allowing us to write the metric as

$$\frac{f'(r_+)^2}{4}R^2\mathrm{d}\tau^2 + \mathrm{d}R^2 + r_+^2\mathrm{d}\Omega^2_{D-2}. \tag{D.39}$$

Note that $R$ approaches 0 as $r$ approaches $r_+$. The first two terms describe a conic metric [41] (i.e. the surface of a cone)

$$ds^2 = \mathrm{d}R^2 + R^2\mathrm{d}\phi^2, \tag{D.40}$$

with the angle given by

$$\phi = \frac{1}{2}\tau f'(r_+). \tag{D.41}$$

Like we said, this is the metric of a conical surface. It presents a singularity unless we identify $\phi = 0$ with $\phi = 2\pi$ . Otherwise the cone is not closed and has two non-connected sides with a volume outside the manifold between them. To solve this we impose the periodicity

$$\phi \in [0, 2\pi]/\sim . \tag{D.42}$$

The symbol "$\sim$" here means that we identify the endpoints of the integral as one. The above result leads to the coordinate $\tau$ being periodic with period $4\pi/f'(r_+)$.

A direct evaluation of the surface gravity yields

$$\kappa = \frac{1}{2}f'(r_+), \tag{D.43}$$

so the inverse of the temperature is given by

$$\beta = \frac{1}{T} = \frac{4\pi}{f'(r_+)}. \tag{D.44}$$

Hence, the coordinate $\tau$ has a period $\beta$,

$$\tau \in [0, \beta] \tag{D.45}$$

Obviously when the black hole is rotating the metric will have a extra term proportional to $(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t)$ and, in the five dimensions case, to $\mathrm{d}t\mathrm{d}\psi$. However, the argument above stands since the first two terms which present the conical singularity can still be handled using the same technique. It is simply a question of re-defining $\mathrm{d}\Omega^2_{D-2}$ to incorporate the extra terms resulting from the reduction of the spherical symmetry to an axial symmetry.

There are actually more fundamental reasons – ones which are beyond the scope of this thesis – why the period of the Euclidean time component is related to the temperature. This comes from the path-integral formulation of quantum gravity and an underlying relation between the action of the quantum theory and the partition function in Statistical Mechanics [24].

# Appendix E

# Chern-Simons Term in Electrodynamics

The classical electrodynamics Lagrangian density is given by ([11])

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{ab}F_b - A_a J^a, \tag{E.1}$$

with the field strength tensor defined by

$$F_{ab} = \partial_a A_b - \partial_b A_a. \tag{E.2}$$

We could of course have used covariant derivatives in the above expression but we are not worried about coupling our electromagnetic fields to gravity at the moment, so it does not matter. The Euler-Lagrange equations of motion give

$$\partial_b F^{ab} = J^a. \tag{E.3}$$

Combined with the anti-symmetry of $F_{ab}$, this leads to the usual current conservation discussed in §C.1.1:

$$\partial_a J^a = 0. \tag{E.4}$$

The theory given by (E.1) is a sensible theory in that it is gauge invariant, Lorentz invariant, and local. The last property of course implies that the theory does not break causality. In our usual 3+1 dimensions there is nothing much to add to the discussion. However, in 2+1 dimensions we have an interesting feature: we can define another Lagrangian called the **Chern-Simons Lagrangian** by ([61])

$$\mathcal{L}_{\text{CS}} = \frac{\kappa}{2}\epsilon^{abc}A_a\partial_b A_c - A_a J^a. \tag{E.5}$$

The theory given by the Chern-Simons Lagrangian is also gauge-invariant, Lorentz-invariant and local [61]. In that sense it is a sensible theory. Now notice that the Lagrangian part given by $\epsilon^{abc}A_a\partial_b A_c$ can only exist in 3 dimensions; in 4 dimensions the indices simply do not work out.

The Chern-Simons action in 2+1 dimensions is of course simply the integration of that Lagrangian

$$S_{\text{CS}} = \frac{\kappa}{2}\int \mathrm{d}^3x\epsilon^{abc}A_a\partial_b A_c, \tag{E.6}$$

which at first does not seem gauge-invariant since it depends explicitly on $A$. However, under the gauge transformation $A_a \to A_a + \partial_a f$, the variation of the Lagrangian is given by

$$\delta\mathcal{L}_{\text{CS}} = \frac{\partial\mathcal{L}}{\partial A_a}\delta A_a + \frac{\partial\mathcal{L}}{\partial\left(\partial_b A_a\right)}\delta\left(\partial_b A_a\right). \tag{E.7}$$

The first variation is given by

$$\delta A_a = \partial_a f, \tag{E.8}$$

and the second is given by

$$\delta\left(\partial_b A_a\right) = \partial_b\partial_a f. \tag{E.9}$$

Since partial derivatives commute, the term $\epsilon^{abc}\partial_b\partial_a f$ will vanish. So the variation of the Lagrangian is given by

$$\begin{aligned}\delta\mathcal{L}_{\text{CS}} &= \frac{\kappa}{2}\epsilon^{abc}\left(\partial_a f\right)\partial_b A_c + J^a\partial_a f\\ &= \frac{\kappa}{2}\epsilon^{abc}\partial_a f\partial_b A_c - \left(\partial_a f\right)J^a.\end{aligned} \tag{E.10}$$

We now use the fact that current is conserved, i.e. that $\partial_a J^a = 0$, to re-write the above expression as a total derivative

$$\begin{aligned}\delta\mathcal{L}_{\rm CS} = \delta\mathcal{L}_{\rm CS} &= \frac{\kappa}{2}\epsilon^{abc}\partial_a f \partial_b A_c - \partial_a\left(f J^a\right) \\ &= \partial_a\left(\frac{\kappa}{2}\epsilon^{abc}\partial_a f\partial_b A_c - f J^a\right). \end{aligned} \tag{E.11}$$

So the variation of the action is evidently given by a surface term which, in many situations, can be taken to 0 [62]. Direct application of Euler-Lagrange equation leads to the following equation of motion ([61])

$$\frac{\kappa}{2}\epsilon^{abc}F_{bc} = J^a. \tag{E.12}$$

It is easy to see that equation (E.12) leads to a divergenceless current as well.

A similar theory can be defined in any odd dimensions of spacetime. For instance, in 5 dimensions we can define the Chern-Simons Lagrangian by

$$\mathcal{L} = \epsilon^{abcde}A_b\partial_b A_c\partial_d A_e. \tag{E.13}$$

We conclude that the most general electrodynamics theory in odd dimensions then is formed by the superposition of the Maxwell and Chern-Simons Lagrangians.

When we provided the solution for a charged black hole in four dimensions in §1.4 we said that it was found by constraining the metric to satisfy Maxwell's equations in addition to the Einstein equation. For odd-dimensional spacetimes however, *the most general* solution for a charged black hole should solve the Maxwell-Chern-Simons equations of motion resulting from the variation of the sum $\mathcal{L}_{\rm Maxwell} + \mathcal{L}_{\rm CS}$.

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